Theory of Elasticity for Scientists and Engineers
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Theory of Elasticity for Scientists and Engineers

With 110 Illustrations
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Preface

This book is intended to be an introduction to elasticity theory. It is assumed that the student, before reading this book, has had courses in mechanics (statics, dynamics) and strength of materials (mechanics of materials). It is written at a level for undergraduate and beginning graduate engineering students in mechanical, civil, or aerospace engineering. As a background in mathematics, readers are expected to have had courses in advanced calculus, linear algebra, and differential equations. Our experience in teaching elasticity theory to engineering students leads us to believe that the course must be problem-solving oriented. We believe that formulation and solution of the problems is at the heart of elasticity theory.1 Of course orientation to problem-solving philosophy does not exclude the need to study fundamentals. By fundamentals we mean both mechanical concepts such as stress, deformation and strain, compatibility conditions, constitutive relations, energy of deformation, and mathematical methods, such as partial differential equations, complex variable and variational methods, and numerical techniques.

We are aware of many excellent books on elasticity, some of which are listed in the References. If we are to state what differentiates our book from other similar texts we could, besides the already stated problem-solving orientation, list the following: study of deformations that are not necessarily small, selection of problems that we treat, and the use of Cartesian tensors only. Why did we use only Cartesian tensors? First, we believe that it is easier for the readership that we had in mind to accept Cartesian tensor formalism as an introduction to tensor methods in mechanics. Second, use of non-cartesian concepts “complicates the task”.2 For the treatment of elasticity theory where both classical and tensor analysis on manifolds is used, the book of Marsden and Hughes3 is recommended. By introducing the nonlinear deformation tensor we are able to treat, systematically, geometrically nonlinear plate theory (von Kármán) and to present elements of stability analysis for elastic bodies. Also we believe that we present,

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1See P. R. Halmos, The Heart of Mathematics. Amer. Math. Monthly. 87, 519-524 (1980) who states, “What mathematics really consists of is problems and solutions. We believe that the same, even in stronger form and more justly, applies to mechanics in general and to the theory of elasticity in particular.”


in a rather simple manner, difficult, very important, problems of contact stresses and stability, usually omitted in introductory courses.

Chapter 1 treats the theory of stress. Euler-Cauchy's principle is introduced and Cauchy's theorem is proved. Equations of equilibrium and motion (by the use of D'Alembert's principle) are obtained in terms of stress components. The rule of transformation of the stress matrix is derived and it is shown that the stress matrix satisfies the transformation law of the second-order Cartesian tensor. Principal stresses and principal directions are obtained and stress tensor invariants introduced. The chapter ends with the classification of stress states at the point (linear, plane, and three-dimensional), based on the properties of the stress tensor.

Chapter 2 begins with the description of deformation at the point. The measure of deformation is introduced as half of the difference of the squares of the deformed and undeformed line element. From this definition the nonlinear strain tensor is obtained and the geometrical meaning of its components are examined. The small strain tensor is derived by linearization and its properties examined. Compatibility conditions for the linear strain tensor are obtained and comment on the compatibility conditions for the nonlinear strain tensor is made. Chapter 2 ends with the description of strain gauges and the problem of determining principal values of the strain tensor from the strain gauge measurements.

These two preliminary chapters that treat the basic concepts of stress and strain are carefully prepared in a rather simple manner, using only Cartesian vector and tensor notation, which will appeal to students and modern readers. For the researcher they offer cylindrical and spherical coordinate expressions of the equations of equilibrium and the strain-displacement relations.

Chapter 3 introduces constitutive equations for an elastic body. It treats the relationship between stress and strain and begins with some historical background, which is likely to be of particular interest to the student. Once again cylindrical and spherical coordinate expressions are given, providing a useful reference for a researcher. Linearization of the constitutive equation at zero deformation is performed and the transform rule for the fourth-order elasticity tensor obtained. Through systematic reduction the forms of constitutive equations for anisotropic, monoclinic, orthotropic, tetragonal, cubic, and isotropic materials are obtained. The thermoelastic stress-strain relation is also presented. Finally compatibility equations in terms of stresses (Beltrami-Michell equations) are derived and a comment concerning other forms of (nonlinear) stress-strain relations for elastic bodies is made. In this chapter the thermal stresses are also included.

Chapter 4 uses the results obtained in the first three chapters to set up the equations of motion as boundary value problems. In this chapter we give a summary of equations for linear elasticity theory and define basic types of boundary value problems for these equations. The practical matter of strategies for obtaining closed-form solutions to the governing equations
of motion is addressed through the use of stress functions, and a number of mathematical approaches are discussed next. The use of scalar and vector potentials in solving elasticity problems is discussed (the Lamé potential, the Galerkin vector, the Love function, the solution method of Papkovich and Neuber, etc.). This chapter ends with the Saint-Venant principle and comments on its derivation.

Having obtained the equations that govern the theory of elasticity, we turn in Chapter 5 to the presentation of a large number of classic problems for which closed-form solutions are available. We present solutions of some important problems of elasticity theory. Torsion, bending, and rotation of a prismatic rod are treated in detail. The Boussinesq elementary solutions of the first and the second kind are discussed and their properties are examined. The topic of half spaces loaded by a tangential force and elastic waves is especially interesting. This chapter is the heart of the book. By collecting these results together, the book offers to the researcher a convenient reference source.

In Chapter 6 we define plane strain, plane stress, and generalized plane stress problems and present the methods of their solution. Specifically, Airy's stress function is introduced, and its solution is discussed. The use of the complex variable method (Kolossov formulas and Goursat representation of a biharmonic function), Fourier transforms, and other procedures are presented and their use is demonstrated on some of the important problems in engineering. Highlights of this chapter include the Muskhelishvili complex variable approach, including its application to the problem of stresses around a crack.

In Chapter 7 we discuss the use of the energy method in elasticity theory. The principles of virtual and complementary work and corresponding variational statements are presented. The use of the energy method in structural mechanics is discussed. Castigliano's theorems are proved and their use is demonstrated.

In Chapter 8 we derive the von Kármán theory of plates. Classical plate theory follows as a special case. Several examples are treated. Also the basic equation of the Reissner-Mindlin theory are presented together with an example.

In Chapter 9 we treat the contact problem for elastic bodies. We present Hertz's solution of the problem and some generalizations of the Hertz solution are mentioned. The problem of elastic impact is also treated in this chapter.

In Chapter 10 we analyze stability of elastic bodies. We introduce the Euler, energy, and dynamic methods of stability analysis and formulate procedures of determining the critical values of parameters for which the equilibrium configuration of an elastic body becomes unstable. Examples illustrating stability analysis are presented.

If Chapter 5 is heart of the book, Chapters 7 through 10 may be thought of as its arms and legs. Each of these chapters contains an advanced topic:
energy methods, plates, dry contact problems, and elastic stability. If the book were used as a one-term text for a first-year graduate course, there might not be enough time to cover the entire text. In such cases the teacher could pick one or two of these last four chapters to round out the course. These four chapters are in the main independent (although Chapter 10 uses results of Chapter 7). At the end of each chapter a selected number of problems are given. Some of the problems extend and generalize the examples treated in the corresponding chapter. They are not always easy ones. Others are examples taken from the exams and are, generally, easier.

We thank many colleagues for their comments and criticisms, especially B. S. Baclic, A. Bajaj, R. Plaut, R. Rand, F. Rimrott, R. Souchet, D. T. Spasic, H. Uberall, and B. D. Vujanovic. In the process of preparing the manuscript we were assisted by Dr. V. Glavardanov, Dr. R. Maretic, Mrs. L. Espeut and Mrs. J. Milidragovic. We thank all of them.

We are indirectly indebted to the founding masters of the modern theory of elasticity, especially Ronald Rivlin and Clifford Truesdell. Their work and those of Stuart Antman, John Ball, Albert Green, Morton Gurtin, Robin Knobs, Ingo Müller, Paul Naghdi, Siavouch Nemat-Nasser, and Walter Noll have had a large influence on our development of the subject.

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Chapter 1

Analysis of Stress

1.1 Introduction

In this section we analyze inner forces in a deformable body. A deformable body could be solid or fluid. We do not go into the mathematical definition of a solid and fluid body\(^1\) (for such a definition, see Truesdell and Noll (1965)). We define a solid as a deformable body that possesses shear strength, i.e., a solid body can support shearing forces (forces that are directed parallel to the material surface on which they act) over the time scale of some natural process or technological application of interest. The fluid body does not have shear strength (cannot sustain shear stresses). In the analysis of stress we are concerned with the study of inner forces and couples that hold a solid body in the shape that it has in the undeformed and deformed states. On the atomic level, those forces are ionic, metallic, and van der Waals forces that act between individual atoms. They hold the atoms in their positions. Such forces could be studied by the methods of solid-state physics. The notion of stress is introduced in continuum mechanics to analyze integrated effects of atomic forces. This is the so-called macroscopic view of the solid body. In it we ignore the discrete nature of the body and assume that the mass of the body is continuously distributed in part of three-dimensional Euclidean space. Therefore when we say point C of a body we are sure that we did not specify the point in the interatomic space and that there is part of the body at this point.

We call a body stress free if the only internal forces that act between points of the body are those necessary to give the body its shape in the absence of all external influences, including gravitational attraction. In other words, we say that the body is stress free if on the atomic level we have only those forces and couples that keep atoms in a given (crystallographic) configuration. Also we exclude the internal forces that are generated during the process of manufacturing of the body to a specific configuration. Thus, a stress-free body is a body in which all the atoms are arranged in their equilibrium interatomic positions. In elasticity theory the term "body is stressed" is reserved for the state of the body in which the acting forces are results of the change of the equilibrium position of the atoms that constitute the body. Those changes of equilibrium positions are results of the

\(^1\) For example, a simple material is solid if there exist a reference configuration such that the isotropy group in this configuration is a subgroup of the orthogonal group.

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deformation of the body. This deformation is a consequence of the action of outer forces and couples. Therefore, the analysis of stress is concerned with the changes of atomic forces (and not with their absolute values) due to the deformation of the body.

The outer forces may be of two kinds: surface and body forces. Body forces are forces acting on all points of the body. Such forces are gravity and inertial forces, for example. Let \( \bar{x}_1, \bar{x}_2, \bar{x}_3 \) denote a fixed system of rectangular Cartesian coordinate axes defined in a configuration \( \kappa \) of the body \( B \). Let \( e_1, e_2, e_3 \) denote the unit vectors along the \( \bar{x}_1, \bar{x}_2, \bar{x}_3 \) axes, respectively. We denote the position vector on an arbitrary point \( M \) of the body \( B \) in \( \kappa \) as

\[
\mathbf{r} = x_1 e_1 + x_2 e_2 + x_3 e_3 = \sum_{i=1}^{3} x_i e_i .
\]

Let \( f_0(x_i) \) be a body force acting on a unit mass of a body in the neighborhood of a body point \( M \). The mass of a part of the body within the volume \( dV \) is equal to \( dm = \rho_0 dV \), where \( \rho_0 \) is the mass density. Therefore the force acting on the part of the body with the volume \( dV \) is \( f_0(x_i) \rho_0 dV \). From this expression we obtain the body force acting on the unit volume of the body in the neighborhood of a point \( M \) in the form \( \rho_0 f_0(x_i) = f(x_i) \).

The sum of all body forces is given as

\[
\mathbf{F}_f = \iiint_{V} \rho_0 f_0(x_i) dV = \iiint_{V} f(x_i) dV .
\]

In (2) \( V \) is the volume of \( B \) in \( \kappa \). The moment of body forces about the point \( O \) (see Fig. 1) becomes

\[
\mathbf{M}_f = \iiint_{V} \mathbf{r} \times f(x_i) dV .
\]

In (3) we used \( \times \) to denote the vector product. The other kind of forces, the surface forces, act on the surface \( S \) of the body (see Fig. 1). They could be uniformly distributed or dissipated. Let \( \mathbf{p} \) be the force acting on a unit area of the surface of the body \( S \). Then, the resultant force and the resultant moment of the surface forces about the point \( O \) are

\[
\mathbf{F}_S = \int_{S} \mathbf{p} dS; \quad \mathbf{M}_S = \int_{S} \mathbf{r} \times \mathbf{p} dS .
\]

If the body \( B \) is in mechanical equilibrium in the configuration \( \kappa \) then we have the following equations of equilibrium

\[
\mathbf{F}_f + \mathbf{F}_S = 0; \quad \mathbf{M}_f + \mathbf{M}_S = 0 .
\]
In (3) and (4) all torque arise from forces. We may remove this restriction by allowing a field of assigned couples. This field may be prescribed with intensity $\mathbf{m}_V$ in $V$ and intensity $\mathbf{m}_S$ on $S$. Then, equations (3) and (4) become

$$\mathbf{M}_f = \int_V \int [\mathbf{m}_V + \mathbf{r} \times \mathbf{f}(x_i)]dV; \quad \mathbf{M}_S = \int_S [\mathbf{m}_S + \mathbf{r} \times \mathbf{p}]dS. \quad (6)$$

The materials in which prescribed couples are acting are called the polar materials.

### 1.2 Stress vector. Cauchy’s theorem

To determine the influence of one part of the body on the other we imagine that we cut the body by a surface so that two parts with the volume $V_1$ and $V_2$ are defined. In continuum mechanics the following axiom of Euler and Cauchy (see Truesdell and Toupin (1960)) is introduced.

*Upon any surface (real or imaginary) that divides the body, the action of one part of the body on the other is equivalent (equipollent) to the system of distributed forces and couples on the surface dividing the body.*

We assume that the body $B$ is in equilibrium. Suppose that, as shown in Fig. 1, we cut the body into two parts. Suppose further that we remove the part $V_2$ and analyze the part $V_1$ only. Since the body as a whole was in equilibrium it will be in equilibrium under the action of surface and body forces that act on part $V_1$ and the system of distributed forces and couples that act on the surface $S$. To describe this uniformly distributed system of forces and couples we fix an arbitrary point $P$ on the surface $S$. Let $\mathbf{n}$ denote the unit normal on $S$ at $P$. Let us consider an element of $S$ with the area $\Delta A$, containing $P$. The element of area $\Delta A$ is assumed to shrink in size toward zero but in such a way that it always contains $P$ and that
its normal is \( n \). Then, we define a quantity \( p_n \) by the relation

\[
p_n = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A},
\]

In (1) the force \( \Delta F \) represents the resultant of all forces acting on the surface element of area \( \Delta A \) that are the result of the action of part \( V_2 \) on the part \( V_1 \). The limit (1) is assumed to exist and is called the stress vector at the point \( P \) for a surface element (of area \( \Delta A \)) whose orientation is specified by the unit vector \( n \). Similarly, we define

\[
m_n = \lim_{\Delta A \to 0} \frac{\Delta M}{\Delta A},
\]

where we used \( \Delta M \) to denote the resultant of all couples acting on the element of the surface of area \( \Delta A \). In his investigations of crystals Voight in 1887 studied a model of an elastic body in which atoms act on each other by central forces and by couples so that \( \Delta M \neq 0 \). Later in 1909, Voight’s theory was extended by E. and F. Cosserat. In the classical theory of elasticity that we present here it is assumed that

\[
m_n = 0,
\]

is satisfied for each point \( P \) and for all \( n \).

From the very definition it is obvious that \( p_n \) is not a vector field in the usual sense, since it depends on both the point \( P \) and the direction, characterized by \( n \). The vector \( p_n \) changes even in the case when the point \( P \) is fixed. For example, suppose that we change \( n \) into \( -n \). Thus, we consider the influence of the part \( V_1 \) of the body on the part \( V_2 \). Then, we have

\[
p_{-n} = \lim_{\Delta A \to 0} \frac{-\Delta F}{\Delta A} = -p_n .
\]

The quantity \(-\Delta F\) represents the influence of the part \( V_1 \) on part \( V_2 \) of the body and is the consequence of Newton’s third law (action equals reaction).

Equation (4) represents Cauchy’s lemma: the stress vectors acting upon opposite sides of the same surface are equal in magnitude and opposite in direction.

The set of vectors \( p_n = p_n(P) \) at the point \( P \) for all unit vectors \( n \) (there are infinitely many such vectors) determines the state of stress at the point \( P \). Cauchy’s stress principle now could be stated as: the inner forces at the point \( P \) of a deformable body are equivalent to the state of stress at the point \( P \).

We now give an example of determining the stress vector. Consider the straight prismatic rod shown in Fig. 2. On its ends the rod is loaded with the forces of intensity \( F \) and \(-F\). Suppose that the cross-sectional area of the rod (for the cross-section perpendicular to the rod axis) is equal to \( A \).
Suppose further that the forces $\Delta F$ are uniformly distributed across $A$ (this assumption is satisfactory in certain cases). Then, we have

$$p_n = \frac{F}{A},$$

(5)

for every point of $A$. If we fix the cross-section with the unit normal $n^*$, then

$$p_{n^*} = \frac{F}{A^*}.$$  

(6)

From (6) we conclude that $p_n$ has an arbitrary orientation with respect to $n$. Thus, in general, we can decompose $p_n$ into directions of $n$ and the direction orthogonal to $n$, that we denote by $t$. Thus,

$$p_n = \sigma_n n + \sigma_t t.$$  

(7)

Since $n$ and $t$ are unit orthogonal vectors, we have (see Fig. 3)

$$|p_n|^2 = \sigma_n^2 + \sigma_t^2.$$  

(8)

The component $\sigma_n$ is called normal stress and the component $\sigma_t$ is called shear stress.
As we stated, the infinitely many vectors $p_n$ determine the stress state at the point. We show next that all those vectors are not independent and that all of them can be determined if the stress vectors on coordinate planes of a rectangular Cartesian coordinate system $\tilde{x}_i$ are known.

We fix a small element of the body at $P$ with sides of length $dx_i$. The stress vector on the plane $dx_i = \text{const.}$ we denote by $p_i$. We assume that the volume element $dV = dx_1 dx_2 dx_3$ belongs to the body in the stressed state. Suppose we fix $p_3$ (see Fig. 4). Then, we can decompose it along the axes $\tilde{x}_i$ so that

$$p_3 = \sum_{k=1}^{3} (p_3)_k e_k = (p_3)_k e_k.$$  \hspace{1cm} (9)

The quantities $(p_3)_k$ represent the component of the vector $p_3$ along the $\tilde{x}_k$ axis. From now on we write $(p_i)_j = \sigma_{ij}$.

![Figure 4](image)

Then, in general, from equation (9) we obtain

$$p_i = \sigma_{ij} e_j.$$  \hspace{1cm} (10)

If we multiply (10) by $e_k$ and use the fact that the system $e_k$ is an orthogonal system of unit vectors, we obtain

$$\sigma_{ij} = p_i \cdot e_j.$$  \hspace{1cm} (11)

The stress vectors for three coordinate planes, that is, $p_i$, $i = 1, 2, 3$, define through (11) nine quantities $\sigma_{ij}$, $i, j = 1, 2, 3$. These quantities could be arranged as elements of a 3 by 3 matrix $[\sigma_{ij}]$ in the following way

$$[\sigma_{ij}] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}.$$  \hspace{1cm} (12)

\footnote{In writing (9) we observed the Einstein repeated indices convention: repeated indices are summed from $k = 1$ to $k = 3$.}
The first index in \( \sigma_{ij} \) denotes the coordinate plane on which the stress vector acts and the second index denotes the axis on which the stress vector is projected. In Fig. 5 we show components of the stress vectors for coordinate planes.

In drawing the figure the following sign convention is observed.

a) The components \( \sigma_{11}, \sigma_{22}, \) and \( \sigma_{33} \) are considered positive if they coincide with the outer normal to the corresponding plane.

b) The components \( \sigma_{12}, \sigma_{13}, \sigma_{21}, \sigma_{23}, \sigma_{31}, \) and \( \sigma_{32} \) are positive if they project along the positive axes \( \bar{x}_i, \bar{x}_2, \) or \( \bar{x}_3 \) and the plane on which they act has a positive outer normal.

The elements \( \sigma_{11}, \sigma_{22}, \) and \( \sigma_{33} \) are normal stresses and \( \sigma_{12}, \sigma_{13}, \sigma_{21}, \sigma_{23}, \sigma_{31}, \) and \( \sigma_{32} \) are shear stresses, according to our definition.

We show next that the matrix \([\sigma_{ij}]\) determines completely the stress state at the point. Consider an infinitesimal tetrahedron positioned at the point \( P \) where we want to determine the stress state (see Fig. 6).

The tetrahedron PMNK is chosen in such a way that its points MN and K are on the \( \bar{x}_1, \bar{x}_2, \) and \( \bar{x}_3 \) axes, respectively. Let the area of the face KMN be \( dA \). The unit normal on the face KMN we denote by \( \mathbf{n} \). The stress vector on the face KMN could be written as

\[
\mathbf{p}_n = (\mathbf{p}_n)_i \mathbf{e}_i .
\]  

The unit vector \( \mathbf{n} \) we write in the system \( \bar{x}_i \) as

\[
\mathbf{n} = n_i \mathbf{e}_i ,
\]  

where the components of \( \mathbf{n} \) are given as

\[
n_i = \cos \angle(\mathbf{n}, \mathbf{e}_i) .
\]
We note that $n_i$ satisfy $n_1^2 + n_2^2 + n_3^2 = 1$. Also, if we introduce the angles $\alpha$, $\beta$, and $\gamma$ (see Fig. 7), then

$$n_1 = \cos \alpha; \quad n_2 = \cos \beta; \quad n_3 = \cos \gamma.$$  \hfill (16)

Let $dA_i$ be the area of the tetrahedron face whose normal is the $\bar{x}_i$ axis (e.g., $dA_1$ is the area of KPN). Then, the areas of the tetrahedron faces are given by the following expressions

$$dA_i = n_i dA.$$  \hfill (17)

Note also that the volume of the tetrahedron is $dV = Ah/3$, where $h$ is its altitude. The tetrahedron is in equilibrium under the action of the following surface forces

$$p_n dA; \quad - p_1 dA_1; \quad - p_2 dA_2; \quad - p_3 dA_3.$$  \hfill (18)

Also, the body force of intensity $f$ per unit volume is acting. Then, the resultant force on the element of volume $dV$ is given by

$$dF = fdV.$$  \hfill (19)

The tetrahedron will be in equilibrium if the following condition (sum of all forces) holds

$$p_n dA - p_1 dA_1 - p_2 dA_2 - p_3 dA_3 + dF = 0.$$  \hfill (20)
By multiplying (20) with the unit vector $e_1$ for example, we obtain the following relation

$$(P_n)_{1} - \sigma_{11}dA_1 - \sigma_{21}dA_2 - \sigma_{31}dA_3 + f_1dAh/3 = 0. \quad (21)$$

By using (17) in (21) and by taking the limit when $h \to 0$, we obtain

$$(P_n)_{1} = \sigma_{11}n_1 + \sigma_{21}n_2 + \sigma_{31}n_3 = \sigma_{11}\cos\alpha + \sigma_{21}\cos\beta + \sigma_{31}\cos\gamma = \sigma_{k1}n_k. \quad (22)$$

Similar results are obtained when (20) is multiplied by $e_2$ and $e_3$ so that, in general, we have

$$(P_n)_{i} = \sigma_{ji}n_j. \quad (23)$$

Equation (23) expresses Cauchy’s fundamental theorem: from the stress vectors acting across three mutually perpendicular planes at a point all stress vectors are determined. They are given by (23) as a linear function of $\sigma_{ij}$.

Note that the surface force $p$ used in (1.1-4) could be expressed as

$$(p)_{i} = (P_n)_{i} = \sigma_{ji}n_j, \quad (24)$$

where $n$ is the outer normal to the surface $S$ of $B$ in $\kappa$.

Since $\sigma_{ij}$ are given in the form of a matrix (12) and since in finitely dimensional vector spaces linear transformations are represented by matrices, it follows that Cauchy’s theorem could be expressed as: the stress vector at the point is a linear transformation of the unit normal of the plane for which the stress vector is determined. In matrix form (23) could be expressed as

$$P_n = \begin{bmatrix} (P_n)_{1} \\ (P_n)_{2} \\ (P_n)_{3} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{21} & \sigma_{31} \\ \sigma_{12} & \sigma_{22} & \sigma_{32} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \sigma^T n, \quad (25)$$
Cauchy's theorem shows that the stress state at the point is completely determined by matrix \([\sigma_{ij}]\). Infinitely many stress vectors (for different \(n\)) are not mutually independent and could be determined from stress vectors corresponding to three coordinate planes.

### 1.3 Equilibrium equations in terms of stress components

Consider an element of the body in the neighborhood of point \(P\) in the deformed state. Suppose that the stress state at \(P\) is characterized by a matrix \([\sigma_{ij}]\). In Fig. 8 we show the element in the form of a parallelepiped with the sides \(dx_1, dx_2,\) and \(dx_3\). Also in Fig. 8 we show the \(\bar{x}_1\) component of all forces acting on the parallelepiped.

Note that we could write the change in a stress component, say \(\sigma_{11}\), as

\[
d\sigma_{11} = \sigma_{11}(x_1 + dx_1, x_2, x_3) - \sigma_{11}(x_1, x_2, x_3) = \frac{\partial \sigma_{11}}{\partial x_1} dx_1 .
\]

The sum of all \(\bar{x}_1\) components of forces acting on the parallelepiped reads

\[
\left[ \sigma_{11} + \frac{\partial \sigma_{11}}{\partial x_1} dx_1 - \sigma_{11}\right] dx_2 dx_3 + \left[ \sigma_{21} + \frac{\partial \sigma_{21}}{\partial x_2} dx_2 - \sigma_{21}\right] dx_1 dx_3
\]

\[
+ \left[ \sigma_{31} + \frac{\partial \sigma_{31}}{\partial x_3} dx_3 - \sigma_{31}\right] dx_1 dx_2 + f_1 dx_1 dx_2 dx_3 = 0 .
\]
From (2) we have
\[ \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} + \frac{\partial \sigma_{31}}{\partial x_3} + f_1 = 0 . \] (3)

Similarly we obtain two more equations, so that the complete set of equilibrium equations reads
\[ \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{31}}{\partial x_3} + f_1 = 0; \quad \frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{32}}{\partial x_3} + f_2 = 0; \]
\[ \frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + f_3 = 0 . \] (4)

Equations (4) could be written as
\[ \sigma_{ij,i} + f_j = 0 , \] (5)

where we used the following notation
\[ \frac{\partial}{\partial x_i} (\cdot) = (\cdot,i) . \] (6)

Equation (5) must hold at each point of the body. For the parallelepiped shown in Fig. 8, the sum of moments of all forces and couples must also be equal to zero. This condition, for the \( x_3 \) axis (the \( x_3 \) axis passes through the mass center of the parallelepiped and is parallel with the axis \( x_3 \)) reads (see Fig. 9)
\[ \left[ \sigma_{12} + \frac{\partial \sigma_{12}}{\partial x_1} dx_1 + \sigma_{12} \right] \frac{dx_2 dx_3 dx_1}{2} \]
\[ - \left[ \sigma_{21} + \frac{\partial \sigma_{21}}{\partial x_2} dx_2 + \sigma_{21} \right] \frac{dx_1 dx_3 dx_2}{2} = 0 . \] (7)

In writing (7) we assumed that there are no distributed body couples. From (7) it follows that
\[ 2\sigma_{12} + \frac{\partial \sigma_{12}}{\partial x_1} dx_1 = 2\sigma_{21} + \frac{\partial \sigma_{21}}{\partial x_2} dx_2 . \] (8)

In the limit when \( dx_1 \to 0 \) and \( dx_2 \to 0 \) from (8) the condition \( \sigma_{12} = \sigma_{21} \) is obtained. Similar expressions we obtain from the moment equations for the other two axes, so that
\[ \sigma_{ij} = \sigma_{ji} . \] (9)

Equation (9) implies symmetry of the stress matrix.3

---

3This result is sometimes called Boltzmann’s axiom. It does not hold for polar materials, i.e., for materials with assigned couples.
Finally with (9) we could write Cauchy’s theorem (1.2-23) and the equilibrium equations (5) in the form

\[(p_n)_i = \sigma_{ij} n_j; \quad \frac{\partial \sigma_{ij}}{\partial x_j} + f_i = 0 . \tag{10}\]

Equations (10) in direct notation read

\[p_n = \sigma n; \quad \text{div} \sigma + f = 0 . \tag{11}\]

Equations (10)_2 are local in the sense that they hold at each point of the body where the stress field is sufficiently smooth. One may pose the following question. If (10)_2 holds do the global equilibrium equations (1.1-5) also hold?

To answer this question we use (1.1-5)_1 and (1.2-24), and obtain the following expression

\[F_f + F_S = \int \int \int_V f dV + \int \int_S \sigma n dS = 0 . \tag{12}\]

The component form of (12) reads

\[\int \int \int_V f_i dV + \int \int_S \sigma_{ij} n_j dS = \int \int \int_V \left[ f_i + \frac{\partial \sigma_{ij}}{\partial x_j} \right] dV = 0 , \tag{13}\]

where we used the Gauss theorem.\(^4\) The expression in brackets on the right-hand side of (13) is equal to zero (this is the equilibrium equation

\(^4\)Let \(b\) be a vector field in the domain \(V\) with the boundary \(S\); let the components \(b_k\).
1.3 Equilibrium equations in terms of stress components

Thus (4) guarantees that the sum of all forces is equal to zero. Next we analyze the sum of all moments (1.2-5). By using Cauchy’s lemma we can write

\[ \int \int \int (r \times f) dV + \int \int (r \times \sigma) dS = 0. \]  

(14)

We analyze the components of (14). For example, the \( \bar{x}_1 \) component of (14) reads

\[ \int \int \int (x_2 f_3 - x_3 f_2) dV + \int \int (x_2 \sigma_{3i} n_i - x_3 \sigma_{2i} n_i) dS = 0. \]  

(15)

The second term in (15) could be transformed as

\[ \int_S (x_2 \sigma_{3i} n_i - x_3 \sigma_{2i} n_i) dS = \int \int \int \left\{ x_2 \left[ \frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} \right] - x_3 \left[ \frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} \right] + \sigma_{32} - \sigma_{23} \right\} dV. \]  

(16)

By using (16) in (15) we obtain

\[ \int \int \int \left\{ x_2 \left[ \frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + f_3 \right] - x_3 \left[ \frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} + f_2 \right] + \sigma_{32} - \sigma_{23} \right\} dV = 0. \]  

(17)

From the equilibrium equations (4) and the symmetry of the stress matrix (9) we conclude that the second equation of global equilibrium (sum of all moments) is satisfied. We note that the problem of deriving (9) could be viewed differently. Namely, we could start from the global expression for the sum of moments of all forces (14) in the form (see Landau and Lifshitz 1987)

\[ \int_S (\sigma_{il} x_k - \sigma_{kl} x_i) n_l dS + \int \int (\sigma_{ki} - \sigma_{ik}) dV = 0. \]  

(18)

From (18) it could be concluded that

\[ \sigma_{ik} - \sigma_{ki} = 2 \frac{\partial}{\partial x_i} \varphi_{iki} \quad \varphi_{ikl} = -\varphi_{kil}. \]  

(19)

of \( b \) be continuous and have continuous derivatives in \( V \). Let \( S \) be a piecewise smooth surface in \( V \) that forms the complete boundary of \( V \) and let \( n \) be the outer normal to \( S \). Then we have

\[ \int_S b_k n_k dS = \int \int (\partial b_k / \partial x_k) dV. \]
where we use $\varphi_{ikl}$ to denote the components of an arbitrary tensor skew symmetric in the first two indices. It could be shown that in the case when $\varphi_{ikl}$ is not zero, the stress components must depend on higher order gradients of the displacement vector. Classical elasticity theory is not concerned with such cases.

1.4 The basic lemma of stress analysis

Consider two planes passing through the same point of an elastic body. Let $\mathbf{n}$ and $\mathbf{m}$ be the unit normal vectors on those planes. The stress vector for each plane is

$$\mathbf{p}_n = \sigma \mathbf{n} ; \quad \mathbf{p}_m = \sigma \mathbf{m} .$$

(1)

If we take the scalar product $\mathbf{p}_n$ with $\mathbf{m}$ we obtain

$$\mathbf{p}_n \cdot \mathbf{m} = \sigma_{ij} n_j m_i = \sigma_{ji} m_i n_j = \mathbf{p}_m \cdot \mathbf{n} ,$$

(2)

where we used the symmetry property of $\sigma_{ij}$. Equation (2) represents the basic lemma of stress analysis: the projection of the stress vector calculated for the plane with unit normal $\mathbf{n}$ on the direction defined by unit vector $\mathbf{m}$, is equal to the projection of the stress vector calculated for the plane with unit normal $\mathbf{m}$, on the direction defined by unit vector $\mathbf{n}$.

Suppose that $\mathbf{n}$ and $\mathbf{m}$ are orthogonal. Suppose further that we decompose $\mathbf{p}_n$ according to (1.2-7). Then

$$\mathbf{p}_n = \sigma_n \mathbf{n} + \sigma_t \mathbf{t} .$$

(3)

By taking the scalar product of $\mathbf{p}_n$ with $\mathbf{m}$ we obtain

$$\mathbf{p}_n \cdot \mathbf{m} = (\sigma_t \mathbf{t}) \cdot \mathbf{m} = \sigma_{nm} ,$$

(4)

since $\mathbf{n}$ and $\mathbf{m}$ are orthogonal. In (4) we used $\sigma_{nm}$ to denote the projection of the stress vector $\mathbf{p}_n$ on the direction $\mathbf{m}$. This notation is in accordance with those introduced in Section 1.2. The basic lemma of stress analysis now implies that

$$\sigma_{nm} = \sigma_{mn} .$$

(5)

The relation (5) expresses the following important fact: the tangential stresses in two mutually perpendicular planes are of equal magnitude. They are both directed either towards the intersection of the planes, or away from it.

1.5 Equilibrium equations in coordinate systems

Consider a part of an elastic body in the neighborhood of an arbitrary point $P$. Let the points of the body be referred to a cylindrical coordinate
1.5 Equilibrium equations in coordinate systems

The stress vectors on the coordinate surfaces $r = \text{const.}$, $\theta = \text{const.}$, and $z = \text{const.}$ are denoted as $\mathbf{p}_r$, $\mathbf{p}_\theta$, and $\mathbf{p}_z$, respectively. Their components are shown in Fig. 10. We decompose stress vectors along the coordinate lines so that

\[
\begin{align*}
\mathbf{p}_r &= \sigma_{rr} \mathbf{e}_r + \sigma_{r\theta} \mathbf{e}_\theta + \sigma_{rz} \mathbf{e}_z ; \\
\mathbf{p}_\theta &= \sigma_{\theta r} \mathbf{e}_r + \sigma_{\theta\theta} \mathbf{e}_\theta + \sigma_{\theta z} \mathbf{e}_z ; \\
\mathbf{p}_z &= \sigma_{z r} \mathbf{e}_r + \sigma_{z\theta} \mathbf{e}_\theta + \sigma_{zz} \mathbf{e}_z .
\end{align*}
\]

From Fig. 11 we conclude that the forces acting on the elementary surfaces of the body element are

\[
\begin{align*}
\mathbf{S}_r &= \mathbf{p}_r rd\theta dz; \\
\mathbf{S}_\theta &= \mathbf{p}_\theta rdr dz; \\
\mathbf{S}_z &= \mathbf{p}_z r d\theta dr.
\end{align*}
\]

The body force of intensity $\mathbf{f}$ is also acting on the elementary part shown in Fig. 11. With the notations of equation (2) the equilibrium condition (sum of all forces equal to zero) leads to

\[
\frac{\partial \mathbf{S}_r}{\partial r} dr + \frac{\partial \mathbf{S}_\theta}{\partial \theta} d\theta + \frac{\partial \mathbf{S}_z}{\partial z} dz + \mathbf{f} r d\theta dr dz = 0 .
\]

Figure 10

Using (2) in (3) and taking into account the relations\(^5\)

\[
\begin{align*}
\frac{\partial \mathbf{e}_r}{\partial \theta} &= \mathbf{e}_\theta ; \\
\frac{\partial \mathbf{e}_\theta}{\partial \theta} &= -\mathbf{e}_r ; \\
\frac{\partial \mathbf{e}_z}{\partial \theta} &= 0 ,
\end{align*}
\]

\(^5\)The equation (4) can be proved if we express $\mathbf{e}_r$, $\mathbf{e}_\theta$, and $\mathbf{e}_z$ in terms of $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ ($\mathbf{e}_r = \cos \theta \mathbf{e}_1 + \sin \theta \mathbf{e}_2$; $\mathbf{e}_\theta = -\sin \theta \mathbf{e}_1 + \cos \theta \mathbf{e}_2$; $\mathbf{e}_z = \mathbf{e}_3$) and so obtained expressions differentiate.
we obtain

\[
\left( \frac{\partial \sigma_{rr}}{\partial r} e_r + \frac{\partial \sigma_{r\theta}}{\partial r} e_\theta + \frac{\partial \sigma_{rz}}{\partial z} e_z + \sigma_{r\theta} e_r + \sigma_{r\theta} e_\theta + \sigma_{rz} e_z \right) r d\theta dz dr + (\sigma_{rr} e_r + \sigma_{r\theta} e_\theta + \sigma_{rz} e_z) d\theta dz dr
\]

\[\quad + \left( \frac{\partial \sigma_{r\theta}}{\partial \theta} e_r + \frac{\partial \sigma_{\theta\theta}}{\partial \theta} e_\theta + \frac{\partial \sigma_{\theta\theta}}{\partial \theta} e_\theta - \sigma_{\theta\theta} e_\theta + \frac{\partial \sigma_{\theta z}}{\partial z} e_z \right) dr dz d\theta \]

\[\quad + \left( \frac{\partial \sigma_{rz}}{\partial z} e_r + \frac{\partial \sigma_{z\theta}}{\partial z} e_\theta + \frac{\partial \sigma_{zz}}{\partial z} e_z \right) r d\theta dz dr
\]

\[\quad + (f_r e_r + f_\theta e_\theta + f_z e_z) r d\theta dz dr = 0. \quad (5)\]

From (5) it follows that

\[
\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{1}{r} (\sigma_{rr} - \sigma_{\theta\theta}) + f_r = 0; \]

\[
\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2}{r} \sigma_{\theta r} + f_\theta = 0; \]

\[
\frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{z\theta}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{1}{r} \sigma_{rz} + f_z = 0. \quad (6)\]

Using the condition that the moment of all forces acting on the element shown in Fig. 11 is equal to zero, we obtain

\[
\sigma_{r\theta} = \sigma_{\theta r}; \quad \sigma_{r z} = \sigma_{z r}; \quad \sigma_{\theta z} = \sigma_{z \theta}. \quad (7)\]

Similarly we obtain the equilibrium equations in the spherical coordinate system \((\rho, \theta, \varphi)\) shown in Fig. 12.
The unit vectors $e_\rho$, $e_\theta$, and $e_\varphi$ and their derivatives are connected through the following relations

$$
e_\rho = \sin \varphi \cos \theta e_1 + \sin \varphi \sin \theta e_2 + \cos \varphi e_3 ;$$

$$e_\varphi = \cos \varphi \cos \theta e_1 + \cos \varphi \sin \theta e_2 - \sin \varphi e_3 ;$$

$$e_\theta = -\sin \theta e_1 + \cos \theta e_2 ; \quad \frac{\partial e_\varphi}{\partial \rho} = \frac{\partial e_\varphi}{\partial \rho} = \frac{\partial e_\theta}{\partial \rho} = 0 ;$$

$$\frac{\partial e_\varphi}{\partial \varphi} = -e_\rho ; \quad \frac{\partial e_\rho}{\partial \varphi} = e_\varphi ; \quad \frac{\partial e_\theta}{\partial \varphi} = 0 ;$$

$$\frac{\partial e_\theta}{\partial \theta} = -\sin \varphi e_\rho - \cos \varphi e_\varphi ; \quad \frac{\partial e_\varphi}{\partial \theta} = \sin \varphi e_\theta ; \quad \frac{\partial e_\rho}{\partial \theta} = \cos \varphi e_\theta . \quad (8)$$

Using (8) and a procedure similar to the previous one we obtain the equilibrium equations in the spherical coordinate system as

$$\frac{\partial \sigma_{\rho \rho}}{\partial \rho} + \frac{1}{\rho} \frac{\partial \sigma_{\rho \varphi}}{\partial \varphi} + \frac{1}{\rho \sin \varphi} \frac{\partial \sigma_{\rho \theta}}{\partial \theta} + \cos \varphi \frac{\partial \sigma_{\rho \rho}}{\rho \sin \varphi} - \frac{\partial \sigma_{\varphi \varphi}}{\partial \rho} + \frac{\partial \sigma_{\varphi \theta}}{\partial \rho} \sigma_{\varphi \phi} + \frac{1}{\rho} (2\sigma_{\rho \rho} - \sigma_{\varphi \varphi} - \sigma_{\theta \theta}) + f_\rho = 0 ;$$

$$\frac{\partial \sigma_{\varphi \rho}}{\partial \rho} + \frac{1}{\rho} \frac{\partial \sigma_{\varphi \varphi}}{\partial \varphi} + \frac{1}{\rho \sin \varphi} \frac{\partial \sigma_{\varphi \theta}}{\partial \theta} + \frac{3}{\rho} \sigma_{\varphi \rho} + \cos \varphi \frac{\sigma_{\rho \rho}}{\rho \sin \varphi} \sigma_{\varphi \phi} - \sigma_{\theta \theta} + f_\varphi = 0 ;$$

$$\frac{\partial \sigma_{\theta \rho}}{\partial \rho} + \frac{1}{\rho} \frac{\partial \sigma_{\theta \varphi}}{\partial \varphi} + \frac{1}{\rho \sin \varphi} \frac{\partial \sigma_{\theta \theta}}{\partial \theta} + \frac{3}{\rho} \sigma_{\theta \rho} + \frac{2 \cos \varphi}{\rho \sin \varphi} \sigma_{\theta \phi} + f_\theta = 0 . \quad (9)$$

We present now a special case of the system (6). Namely, if the problem under consideration has radial symmetry then

$$\sigma_{r \theta} = 0 . \quad (10)$$
With the condition (10) the system (6) becomes

\[
\frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) + f_r = 0 ;
\]
\[
\frac{1}{r} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + f_\theta = 0 ;
\]
\[
\frac{\partial \sigma_{zr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{z\theta}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{1}{r} \sigma_{zr} + f_z = 0 .
\] (11)

We note that in all preceding expressions the components of the stress vector \(\sigma_{rr}, ..., \sigma_{zz}\) are so-called physical components of the stress vector. They have the dimension [force/area].

1.6 Transformation of stress matrix. Stress tensor

In this section we consider two important questions about the matrix \([\sigma_{ij}]\). First we analyze the transformation of the stress matrix in the case where the coordinate system is subjected to a special type of transformation. Suppose that a Cartesian coordinate system with axes \(x_i, i = 1, 2, 3\) is defined and that the stress matrix in this system has components \(\sigma_{ij}\).

Suppose further that another Cartesian coordinate system \(x_i^*, i = 1, 2, 3\) is given with the same origin (see Fig. 13). We use \(c_{ik}\) to denote the direction cosines, that is

\[
c_{ik} = \cos \angle(x_i^*, x_k) .
\] (1)

With the notation (1) we can express the unit vector along the \(x_i^*\) axis as

\[
e_i^* = \cos \angle(x_i^*, \bar{x}_1)e_1 + \cos \angle(x_i^*, \bar{x}_2)e_2 + \cos \angle(x_i^*, \bar{x}_3)e_3
\]
\[
= c_{i1}e_1 + c_{i2}e_2 + c_{i3}e_3 = c_{ik}e_k .
\] (2)

Note that, because \(e_i^*\) is the unit vector and since \(e_i^*\) is orthogonal to \(e_k^*\) for \(i \neq k\), we have

\[
c_{i1}^2 + c_{i2}^2 + c_{i3}^2 - 1 = 0, \quad c_{i1}c_{k1} + c_{i2}c_{k2} = 0 \quad \text{for } i \neq k .
\] (3)

We could write equation (3) as

\[
c_{im}c_{km} = \delta_{ik} ,
\] (4)

where we used \(\delta_{ik}\) to denote the Kronecker symbol; that is,

\[
\delta_{ik} = \begin{cases} 
1 & \text{if } i = k \\
0 & \text{if } i \neq k 
\end{cases} .
\] (5)
1.6 Transformation of stress matrix. Stress tensor

By using the notation (1) we can express the unit vector $e_i$ in terms of $e_j$ as

$$e_i = c_{1i}e_1^* + c_{2i}e_2^* + c_{3i}e_3^* = c_{ji}e_j^*. \tag{6}$$

If we form a three by three matrix from the elements $c_{ij}$ and denote this matrix by $Q = [c_{ij}]$, then equation (4) shows that $Q$ is an orthogonal matrix; that is, it satisfies $Q^T = Q^{-1}$. We note that for $Q$ orthogonal $\det Q = \pm 1$. Also, if $\det Q = 1$ the matrix is called a proper orthogonal, while if $\det Q = -1$ the matrix is called an improper orthogonal matrix.

Our first task could be stated as: given a stress matrix with elements $\sigma_{ij}$ in the system $\bar{x}$, $i = 1, 2, 3$ (i.e., $[\sigma_{ij}]$) find the elements $\sigma_{ij}^*$ of the same stress matrix in the system $x_i^*$, $i = 1, 2, 3$ (we denote this matrix as $[\sigma_{ij}^*]$).\(^6\)

Thus, we want to find

$$[\sigma_{ij}^*] = \begin{bmatrix}
\sigma_{11}^* & \sigma_{12}^* & \sigma_{13}^* \\
\sigma_{21}^* & \sigma_{22}^* & \sigma_{23}^* \\
\sigma_{31}^* & \sigma_{32}^* & \sigma_{33}^*
\end{bmatrix}. \tag{7}$$

To do this, we note that the elements of the first column in (7) represent the components of the stress vector on the coordinate plane whose normal

\(^6\)To avoid errors when we write a stress matrix we always have to indicate in which coordinate system this matrix is defined. This could be achieved by writing $[\sigma_{ij}]_{\bar{x}}$, where the index on brackets indicates the coordinate system used. As a rule, we omit the index on brackets but we clearly indicate before the matrix sign $[\sigma_{ij}]$ the coordinate system that is used in defining $\sigma_{ij}$. 

Chapter 1 Analysis of Stress

is the $x_i^*$ referred to the $x_i^*$ coordinate system with unit vectors $e_i^*$, $i = 1, 2, 3$. From (1.3-11) and (2) we have

$$p_i^* = \sigma e_i^* = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} c_{1i} \\ c_{2i} \\ c_{3i} \end{bmatrix}, \quad (8)$$

or

$$p_i^* = [\sigma_{11} c_{1i} + \sigma_{12} c_{2i} + \sigma_{13} c_{3i}] e_1 + [\sigma_{21} c_{1i} + \sigma_{22} c_{2i} + \sigma_{23} c_{3i}] e_2 + [\sigma_{31} c_{1i} + \sigma_{32} c_{2i} + \sigma_{33} c_{3i}] e_3 = \sigma_{sm} c_{im} e_s. \quad (9)$$

We determine now the components of $p_i^*$ in the system $x_j^*$. This will give the components $\sigma_{ij}$. Recall that for fixed $i$ the values $\sigma_{ij}$ represent the components of the stress vector on the plane whose normal is the $x_j^*$ axis. Thus, by using (9) and (1.2-25) we obtain

$$\sigma_{ij} = p_i^* \cdot e_j^* = [\sigma_{11} c_{1i} + \sigma_{12} c_{2i} + \sigma_{13} c_{3i}] c_{j1} + [\sigma_{21} c_{1i} + \sigma_{22} c_{2i} + \sigma_{23} c_{3i}] c_{j2} + [\sigma_{31} c_{1i} + \sigma_{32} c_{2i} + \sigma_{33} c_{3i}] c_{j3} = \sigma_{sm} c_{im} c_{jk} \delta_{sk} = c_{im} \sigma_{ms} c_{js}, \quad (10)$$

where $\delta_{sk}$ is the Kronecker symbol (5) and we used the symmetry of the stress matrix $\sigma_{ij} = \sigma_{ji}$. If we write the matrix $Q$ as

$$Q = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}, \quad (11)$$

then (10) becomes

$$\begin{bmatrix} \sigma_{11}^* & \sigma_{12}^* & \sigma_{13}^* \\ \sigma_{21}^* & \sigma_{22}^* & \sigma_{23}^* \\ \sigma_{31}^* & \sigma_{32}^* & \sigma_{33}^* \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \times \begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{bmatrix}. \quad (12)$$

Equation (12) we can express in direct notation as

$$\sigma^* = Q \sigma Q^T. \quad (13)$$

In general any set of nine quantities $\sigma_{ij}$ (two index symbols) referred to the set of axes $\bar{x}$, $i = 1, 2, 3$ and transformed to another set of quantities given by

$$\sigma_{ij}^* = c_{im} \sigma_{ms} c_{js}, \quad (14)$$
1.6 Transformation of stress matrix. Stress tensor

in the new coordinate system \(x_i^*, i = 1, 2, 3\) is called a *Cartesian tensor of the second order*.

The name emphasizes the nature of the first tensors in physics. The components \(\sigma_{ij}, i, j = 1, 2, 3\) measure the tensions inside a deformable body.\(^7\) Thus, from now on, instead of stress matrix we shall use the term stress tensor for \(\sigma\).

Another important question concerning the stress tensor may be stated as: is it possible to find a coordinate system \(x_i^*, i = 1, 2, 3\) (or transformation matrix \(Q\) that transforms axes \(X_i, i = 1, 2, 3\) into axes \(x_i^*, i = 1, 2, 3\)) such that in the system \(x_i^*, i = 1, 2, 3\) the stress tensor has the diagonal elements only?

The requirement that the stress tensor have diagonal elements only (off diagonal elements are equal to zero) means that the stress vector on coordinate planes \(x_i^* = \text{const.}\) is parallel to the unit vector \(e_i^*\). In other words, in the system \(x_i^*, i = 1, 2, 3\) the stress vectors acting on the coordinate planes have the normal stress components only while the shear stresses are equal to zero.

Let \(n\) denote the unit normal of a plane on which the stress vector has the normal component only. For such a plane the stress vector must be parallel to the unit normal; that is,

\[
p_n \parallel n.
\] (15)

From (15) and Fig. 14 we obtain

\[
p_n = \lambda n,
\] (16)

where \(\lambda\) is a proportionality factor. Taking into account equation (1.3-11), we can write (16) in the following form

\[
[\sigma n - \lambda n] = 0.
\] (17)

Expanding (17) we obtain the following system of three linear algebraic equations for determining the components of \(n\)

\[
(\sigma_{11} - \lambda)n_1 + \sigma_{12}n_2 + \sigma_{13}n_3 = 0;
\sigma_{21}n_1 + (\sigma_{22} - \lambda)n_2 + \sigma_{23}n_3 = 0;
\sigma_{31}n_1 + \sigma_{32}n_2 + (\sigma_{33} - \lambda)n_3 = 0.
\] (18)

\(^7\)There is disagreement on the origin of the name. Veblen states that “The convenient though historically not well justified name, *tensor* was introduced by A. Einstein (O. Veblen: *Invariants of Quadratic Forms*, Cambridge University Press, Cambridge 1962). On the other hand Brillouin claims that the name was given by German physicist Voight in his *Lehrbuch der Kristalphysik* (L. Brillouin: *Cours de Physique Theorique-Les tenseurs en mécanique et en élasticité*, Masson et Cie., Paris 1960).
Any unit vector \( \mathbf{n} \) satisfying (17) is called the principal direction of \( \sigma \) (other names are: eigenvector, characteristic vector, proper vector). The scalar \( \lambda \) is called a principal value of \( \sigma \) (other names: eigenvalue, characteristic value, proper value). To determine principal directions and principal values of \( \sigma \) we note that the homogeneous system of linear equations has a nontrivial solution if the determinant of the system is equal to zero; that is,

\[
\begin{vmatrix}
\sigma_{11} - \lambda & \sigma_{12} & \sigma_{13} \\
\sigma_{21} & \sigma_{22} - \lambda & \sigma_{23} \\
\sigma_{31} & \sigma_{32} & \sigma_{33} - \lambda
\end{vmatrix} = 0.
\]  

(19)

Expanding (19), we obtain

\[-\lambda^3 + \hat{I}_1 \lambda^2 - \hat{I}_2 \lambda + \hat{I}_3 = 0.\]  

(20)

where we introduced the following notation

\[
\hat{I}_1 = \sigma_{11} + \sigma_{22} + \sigma_{33} = \text{tr } \sigma;
\]

\[
\hat{I}_2 = \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{11}\sigma_{33} - \sigma_{12}^2 - \sigma_{13}^2 - \sigma_{23}^2
\]

\[
= \det \begin{vmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{vmatrix} + \det \begin{vmatrix} \sigma_{11} & \sigma_{13} \\ \sigma_{13} & \sigma_{33} \end{vmatrix} + \det \begin{vmatrix} \sigma_{22} & \sigma_{23} \\ \sigma_{23} & \sigma_{33} \end{vmatrix}
\]

\[
= \frac{1}{2} [\hat{I}_1^2 - \text{tr}(\sigma^2)];
\]

\[
\hat{I}_3 = \sigma_{11}\sigma_{22}\sigma_{33} + 2\sigma_{12}\sigma_{13}\sigma_{23} - (\sigma_{11}\sigma_{23}^2 + \sigma_{22}\sigma_{13}^2 + \sigma_{33}\sigma_{12}^2)
\]

\[
= \det \sigma.
\]  

(21)

Equation (20) is called the characteristic equation of \( \sigma \). The coefficients \( \hat{I}_k \), \( k = 1, 2, 3 \) are called the principal invariants of \( \sigma \) (see Section 1.8).

The cubic equation (20) has three roots. They are either all real (not necessarily different) or one is real and two are complex conjugate. We show next that all principal values of \( \sigma \) are real.
1.6 Transformation of stress matrix. Stress tensor

Suppose that the principal values of $\sigma$ are complex conjugates $\lambda$ and $\bar{\lambda}$. We show that in the case when $\sigma$ is symmetric ($\sigma = \sigma^T$) this cannot be the case. Let $\mathbf{n}$ and $\bar{\mathbf{n}}$ be the principal vectors corresponding to $\lambda$ and $\bar{\lambda}$, respectively. Then, we have

$$\sigma \mathbf{n} = \lambda \mathbf{n} ; \quad \sigma \bar{\mathbf{n}} = \bar{\lambda} \bar{\mathbf{n}} . \quad (22)$$

Now from the scalar product of (22) first by $\bar{\mathbf{n}}$ and then by $\mathbf{n}$ we subtract the results to obtain

$$\bar{\mathbf{n}} \cdot \sigma \mathbf{n} - \sigma \bar{\mathbf{n}} \cdot \mathbf{n} = (\lambda - \bar{\lambda}) \mathbf{n} \cdot \bar{\mathbf{n}} . \quad (23)$$

Since $\mathbf{n}$ is complex, we can write

$$\mathbf{n} = \mathbf{u} + i \mathbf{v} ; \quad \bar{\mathbf{n}} = \mathbf{u} - i \mathbf{v} , \quad (24)$$

where $\mathbf{u}$ and $\mathbf{v}$ are real vectors and $i = \sqrt{-1}$. The norm of the vector $\mathbf{n}$ is

$$\mathbf{n} \cdot \mathbf{n} = |\mathbf{n}|^2 = \mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} = 1 . \quad (25)$$

By using (25) in (23) we obtain

$$\mathbf{n} \cdot (\sigma^T - \sigma) \bar{\mathbf{n}} = (\lambda - \bar{\lambda}) . \quad (26)$$

Since $\sigma$ is symmetric we conclude that $\lambda = \bar{\lambda}$; that is, $\lambda$ is real. It follows then from (18) that $\mathbf{n}$ is real too. Thus we showed that: a symmetric tensor with real elements has real principal values.

Suppose that we have a solution $\lambda$ of the characteristic equation (20). Let $\mathbf{n}$ be the corresponding principal direction. The components of $\mathbf{n}$ are determined from (18) and

$$n_1^2 + n_2^2 + n_3^2 = 1 , \quad (27)$$

since only two equations in (18) are linearly independent. Suppose now that two principal values $\lambda_n$ and $\lambda_m$ are distinct. Let $\mathbf{n}$ and $\mathbf{m}$ be the corresponding principal directions. Then we have

$$\sigma \mathbf{n} = \lambda_n \mathbf{n} ; \quad \sigma \mathbf{m} = \lambda_m \mathbf{m} , \quad (28)$$

or, with $\lambda_n = \sigma_1$, $\lambda_m = \sigma_2$

$$(\sigma_{11} - \sigma_1) n_1 + \sigma_{12} n_2 + \sigma_{13} n_3 = 0 ;$$
$$\sigma_{21} n_1 + (\sigma_{22} - \sigma_1) n_2 + \sigma_{23} n_3 = 0 ;$$
$$\sigma_{31} n_1 + \sigma_{32} n_2 + (\sigma_{33} - \sigma_1) n_3 = 0 , \quad (29)$$

and

$$(\sigma_{11} - \sigma_2) m_1 + \sigma_{12} m_2 + \sigma_{13} m_3 = 0 ;$$
$$\sigma_{21} m_1 + (\sigma_{22} - \sigma_2) m_2 + \sigma_{23} m_3 = 0 ;$$
$$\sigma_{31} m_1 + \sigma_{32} m_2 + (\sigma_{33} - \sigma_2) m_3 = 0 . \quad (30)$$
By multiplying (29) with \( m_1, m_2, \) and \( m_3 \) and (30) with \( n_1, n_2, \) and \( n_3 \) and by subtracting the results, we get

\[
(\sigma_2 - \sigma_1)(n_1m_1 + n_2m_2 + n_3m_3) = (\sigma_2 - \sigma_1)n \cdot m = 0 .
\]  

(31)

From (31) it follows that \( \mathbf{n} \perp \mathbf{m} \); that is: the principal directions corresponding to distinct principal values of \( \sigma \) are mutually orthogonal.

In the case when all principal values are distinct (e.g., \( \sigma_1 > \sigma_2 > \sigma_3 \)) the corresponding principal directions \( \mathbf{n}, \mathbf{m}, \) and \( \mathbf{k} \) are mutually orthogonal. If we select the axes of the coordinate system \( x'_i \), \( i = 1, 2, 3 \) so that the unit vectors along \( x'_i \) are \( \mathbf{n}, \mathbf{m}, \) and \( \mathbf{k} \) then in the system \( x'_i \), \( i = 1, 2, 3 \) the stress tensor has the diagonal elements only (see Fig. 15). The vectors \( \mathbf{n}, \mathbf{m}, \) and \( \mathbf{k} \) in this case constitute the principal basis.

Figure 15

The element of volume taken in the form of a parallelepiped will have on its faces the stress vectors \( \sigma_1 \mathbf{n}, \sigma_2 \mathbf{m}, \) and \( \sigma_3 \mathbf{k} \). Values \( \sigma_1, \sigma_2, \) and \( \sigma_3 \) are called principal stresses. In the principal basis the stress tensor has the form

\[
\begin{bmatrix}
\sigma_1 & 0 & 0 \\
0 & \sigma_2 & 0 \\
0 & 0 & \sigma_3
\end{bmatrix}.
\]  

(32)

In the case when any two characteristic values of the stress tensor \( \sigma \) are equal, the problem of determining the principal basis requires further analysis.

We begin by considering a class of tensors (larger than symmetric) called the normal tensors. Those are tensors with the property

\[
\mathbf{T} \mathbf{T}^T = \mathbf{T}^T \mathbf{T} .
\]  

(33)

Note that symmetric (\( \sigma = \sigma^T \)) and orthogonal (\( \mathbf{Q}^{-1} = \mathbf{Q}^T \)) tensors are normal tensors. Suppose that \( \lambda_3 \) is a characteristic value (possibly having multiplicity higher than one) of a normal tensor. We may assume that \( \lambda_3 \)
is real (see comment after (21)). Then there exists at least one principal direction, say $n_3$, satisfying

$$Tn_3 = \lambda_3 n_3 .$$  \hspace{1cm} (34)

We define a Cartesian coordinate system $\bar{x}_i, i = 1, 2, 3$ with $n_3$ equal to the unit vector $e_3$ oriented along the $\bar{x}_3$ axis. If $v$ is any vector in the plane $\Pi$ defined by $e_1$ and $e_2$ then we have

$$Tn_3 \cdot v = T_{ij} n_3 j v_i = T^T v \cdot n_3 = \lambda_3 n_3 \cdot v = 0 .$$  \hspace{1cm} (35)

From (35) it follows that $T^T v$ is also a vector in the plane $\Pi$ defined by $e_1$ and $e_2$. Since $v$ is an arbitrary vector it follows that $(T^T)_{31} = (T^T)_{32} = 0$ so that $T$ has the representation

$$[T_{ij}] = \begin{bmatrix} T_{11} & T_{12} & 0 \\ T_{21} & T_{22} & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} .$$  \hspace{1cm} (36)

Next we define the second-order tensor $\bar{T}$ in the space of dimension two with components

$$[\bar{T}_{ij}] = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} .$$  \hspace{1cm} (37)

Then for any vector $u$ in $\Pi$ we have

$$Tu = \bar{T}u + \lambda_3 (u \cdot e_3)e_3 = \bar{T}u .$$  \hspace{1cm} (38)

We assume now that $T$ is symmetric (in our application $T = \sigma$). Then $\bar{T}$ is symmetric and there exists a real $\lambda_2$ (the existence of real $\lambda_2$ is guaranteed by the arguments given before (26) with $\bar{T} = \sigma$) such that

$$\bar{T}u = \lambda_2 u ,$$  \hspace{1cm} (39)

with $u = u_1e_1 + u_2e_2$ in $\Pi$. Note that $u$ is also a principal vector of $T$. To see this, we calculate

$$Tu = u_1 Te_1 + u_2 Te_2 = \lambda_2 u_1e_1 + \lambda_2 u_2e_2 = \lambda_2 u .$$  \hspace{1cm} (40)

Therefore $e_3$ and $u$ are principal directions of $T$ with principal values $\lambda_3$ and $\lambda_2$. We show now that the vector $v = u \times e_3$ is also a principal direction of $T$. To do this, note that

$$Tu \cdot v = T_{ij} u_j v_i = u \cdot T^T v = \lambda_2 u \cdot v = 0 .$$  \hspace{1cm} (41)

From (41) it follows that $T^T v =Tv$ is a vector in $\Pi$ perpendicular to $u$ and $e_3$. Consequently $Tv$ must be parallel to $v$, or

$$Tv = \lambda_1 v ,$$  \hspace{1cm} (42)
for some $\lambda_1$. Note that at the beginning we did not assume anything about the multiplicity of $\lambda_3$ (thus $\lambda_3$ could be equal to $\lambda_1$ or $\lambda_2$). Therefore, with $T$ symmetric we can state the following theorem: \textit{there exist always three mutually orthogonal principal directions independently of any multiplicity of the eigenvalues.}

Of course that does not mean that there cannot be \textit{more} than three principal directions for the symmetric second-order tensor.

Suppose now that we consider the stress tensor $\sigma$. Suppose further that it has a principal value of multiplicity 2, say $\lambda_1 = \lambda_2 \neq \lambda_3$. From the result of the theorem just stated we conclude that, corresponding to $\lambda_1$ there exist two mutually orthogonal principal directions $n_1$ and $n_2$ (in equations (40) and (42) denoted by $u$ and $v$). Any unit vector $n$ in the plane determined by $n_1$ and $n_2$ could be written as $n = (n \cdot n_i)n_i$, $i = 1, 2$. Therefore

$$\sigma n = (n \cdot n_i)\sigma n_i = (n \cdot n_i)\lambda_i n_i = \lambda_i n ; \quad \text{no sum on } i. \quad (43)$$

From (43) it follows that \textit{every} vector in the plane determined by $n_1$ and $n_2$ is a principal vector with the corresponding principal value $\lambda_1 = \lambda_2$.

If all principal values are equal $\lambda_1 = \lambda_2 = \lambda_3$, similar analysis shows that every unit vector is a principal vector.

### 1.7 Extreme properties of principal stresses

We now examine the extreme properties of principal stresses. Suppose we are given a plane with the unit normal $n$. The stress vector for this plane is

$$(p_n)_i = \sigma_{ij}n_j , \quad (1)$$

so that the normal stress becomes (see (1.2-7))

$$\sigma_n = p_n \cdot n = \sigma_{ij}n_i n_j , \quad (2)$$

We want to determine the extreme value of $\sigma_n$. Since $n$ is the unit vector we must find the extreme of (2) subject to the constraint

$$n \cdot n = n_i n_i = 1 . \quad (3)$$

The problem of finding an extreme of (2) subject to (3) belongs to the class of constrained optimization problems. By using the Lagrange multiplier, we form a new function

$$F = \sigma_n - \lambda(n \cdot n - 1) , \quad (4)$$

where $\lambda$ is the (unknown) Lagrange multiplier. Then, from (2) we obtain

$$F = \sigma_{ij}n_i n_j - \lambda(n_1^2 + n_2^2 + n_3^2 - 1) . \quad (5)$$
Differentiating (5) with respect to $n_1$, $n_2$, and $n_3$ and setting the resulting expressions to zero, we get

\[
\begin{align*}
\frac{\partial F}{\partial n_1} &= 2[(\sigma_{11} - \lambda)n_1 + \sigma_{12}n_2 + \sigma_{13}n_3] = 0 ; \\
\frac{\partial F}{\partial n_2} &= 2[\sigma_{21}n_1 + (\sigma_{22} - \lambda)n_2 + \sigma_{23}n_3] = 0 ; \\
\frac{\partial F}{\partial n_3} &= 2[\sigma_{31}n_1 + \sigma_{32}n_2 + (\sigma_{33} - \lambda)n_3] = 0 ,
\end{align*}
\]

as a necessary condition for the extreme of $\sigma_n$. If we compare (6) with equations (1.6-18) we conclude that the condition that $p_n$ is parallel with $n$ is equivalent to the condition that $\sigma_n$ is a stationary value. Thus, we already solved the problem of determining $n$ such that $\sigma_n$ has extreme value. The principal directions $n$, $m$, and $k$ are such that (6) is satisfied. The principal values of the stress tensor, say $\sigma_1$, $\sigma_2$, and $\sigma_3$ are candidates for maximal and minimal values of normal stress at a point. The largest of $\sigma_1$, $\sigma_2$, and $\sigma_3$ is the maximal and the smallest is the minimal value of the normal stress for all possible planes through a given point.

From the analysis presented it follows that in principal planes the normal stresses have extreme (or stationary) values and the shear stresses are equal to zero. This also could be shown differently by using the basic lemma of stress analysis (see Section 1.4). To do this we consider two planes I and II passing through point $P$. Let the angle between these two planes be $da$. Let $n$ and $m$ be the unit vectors orthogonal to the planes I and II, respectively (see Fig. 16).

Suppose that the stress vector on the plane I is

\[
p_n = \sigma_{11}(\alpha)e_1 + \sigma_{12}(\alpha)e_2 + \sigma_{13}(\alpha)e_3 .
\]

In the coordinate system shown in Fig. 16 the vector of the unit normal is

\[
n = e_1 .
\]
On the plane II, the stress vector is given as

\[ \mathbf{p}_m = \left[ \sigma_{11}(\alpha) + \frac{d\sigma_{11}}{d\alpha} \right] \mathbf{e}_1 + \left[ \sigma_{12}(\alpha) + \frac{d\sigma_{12}}{d\alpha} \right] \mathbf{e}_2 + \left[ \sigma_{13}(\alpha) + \frac{d\sigma_{13}}{d\alpha} \right] \mathbf{e}_3 . \] (9)

The unit normal \( \mathbf{m} \) could be written as

\[ \mathbf{m} = \cos(\alpha) \mathbf{e}_1 - \sin(\alpha) \mathbf{e}_2 = \mathbf{e}_1 . \] (10)

By using (7) through (10) in (1.4-2), we obtain

\[ [\sigma_{11}\mathbf{e}_1 + \sigma_{12}\mathbf{e}_2 + \sigma_{13}\mathbf{e}_3] \cdot [\cos(\alpha) \mathbf{e}_1 - \sin(\alpha) \mathbf{e}_2] \]

\[ = \left\{ \left[ \sigma_{11}(\alpha) + \frac{d\sigma_{11}}{d\alpha} \right] \mathbf{e}_1 + \left[ \sigma_{12}(\alpha) + \frac{d\sigma_{12}}{d\alpha} \right] \mathbf{e}_2 + \left[ \sigma_{13}(\alpha) + \frac{d\sigma_{13}}{d\alpha} \right] \mathbf{e}_3 \right\} \cdot \mathbf{e}_1 . \] (11)

However, since

\[ \mathbf{e}_3 = \mathbf{e}_3; \quad \mathbf{e}_2 = \sin(\alpha) \mathbf{e}_1 + \cos(\alpha) \mathbf{e}_2 , \] (12)

we can simplify (11) to get

\[ -2\sigma_{12} = \frac{d\sigma_{11}}{d\alpha} , \] (13)

where we used the fact that \( d\alpha \) small. From (13) it follows that for the plane for which \( \sigma_{11} \) has extreme value the shear stress \( \sigma_{12} \) vanishes. As a matter of fact, the normal stress varies at a rate that is twice the shear stress with opposite sign. Similar analysis (rotation of coordinate system in plane I so that \( \mathbf{e}_2 \) is not changed) shows that

\[ -2\sigma_{13} = \frac{d\sigma_{11}}{d\beta} , \] (14)

where \( \beta \) is the angle of rotation about the \( \bar{x}_2 \) axis. Thus the extreme of \( \sigma_{11} \) is in the plane where \( \sigma_{12} = \sigma_{13} = 0 \).

### 1.8 Invariants of the stress tensor

As we showed, the principal stresses are determined from equation (1.6-20), and they have the property of being extreme. That is, the principal stresses determine the smallest and the largest value of normal stresses at
a given point. Since the smallest and largest values of normal stresses at the point do not depend on the choice of the coordinate system, we reach the following important conclusion: the solutions of equation (1.6-20) must be independent of coordinate system.

In other words if \( \sigma_1, \sigma_2, \) and \( \sigma_3 \) are the principal stresses determined from (1.6-20) and the stress tensor in the new coordinate system \( x'_1, x'_2, x'_3 \) has a representation \([\sigma'_{ij}]\), then the equation that determines the principal values in \( x'_1, x'_2, x'_3 \) that is (see (1.6-19)),

\[
\det \begin{vmatrix}
\sigma'_{11} - \sigma & \sigma'_{12} & \sigma'_{13} \\
\sigma'_{21} & \sigma'_{22} - \sigma & \sigma'_{23} \\
\sigma'_{31} & \sigma'_{32} & \sigma'_{33} - \sigma
\end{vmatrix} = 0 ,
\]

must have as solution \( \sigma_1, \sigma_2, \) and \( \sigma_3 \). Thus, we conclude that the principal stresses are invariants of the tensor \( \sigma \), if \( \sigma \) is subjected to the transformation described by \( Q \) given by (1.6-11). From this fact it follows that any quantity that depends on \( \sigma_1, \sigma_2 \) and \( \sigma_3 \) is also invariant when the change of coordinate system is described by \( Q \).

We show now that the coefficients \( \hat{I}_1, \hat{I}_2, \) and \( \hat{I}_3 \) given by (1.6-21) are invariant. First, we write (1.6-20) in the form

\[
(\sigma - \sigma_1)(\sigma - \sigma_2)(\sigma - \sigma_3) = 0 .
\]  

From (2) it follows that

\[
-\sigma^3 + (\sigma_1 + \sigma_2 + \sigma_3)\sigma^2 - (\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3)\sigma + \sigma_1\sigma_2\sigma_3 = 0 .
\]  

By comparing (3) and (1.6-20) we conclude that

\[
\hat{I}_1 = \sigma_{11} + \sigma_{22} + \sigma_{33} = \sigma_1 + \sigma_2 + \sigma_3 ;
\]

\[
\hat{I}_2 = \sigma_{11}\sigma_{22} + \sigma_{11}\sigma_{33} + \sigma_{22}\sigma_{33} - \sigma_{12}^2 - \sigma_{13}^2 - \sigma_{23}^2 \\
= \sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3 ;
\]

\[
\hat{I}_3 = \sigma_{11}\sigma_{22}\sigma_{33} + 2\sigma_{12}\sigma_{13}\sigma_{23} - (\sigma_{11}\sigma_{23}^2 + \sigma_{22}\sigma_{13}^2 + \sigma_{33}\sigma_{12}^2) \\
= \sigma_1\sigma_2\sigma_3 .
\]

It is clear now why in (1.6-21) we called \( \hat{I}_1, \hat{I}_2, \) and \( \hat{I}_3 \) the principal invariants of \( \sigma \).

It could be shown that the stress tensor has only three independent invariants. Every other invariant can be expressed in terms of \( \hat{I}_1, \hat{I}_2, \) and \( \hat{I}_3 \).

### 1.9 Extreme values of shear stresses

We consider now the problem of determining the extreme values of shear stresses at a given point. From (1.2-7) we have the expression for the shear
stress for the plane with the normal \( \mathbf{n}^* \) as

\[
\sigma_i^2 = |p_{n*}|^2 - \sigma_{n*}^2. \tag{1}
\]

Suppose that we know the principal values and principal directions of the tensor \( \mathbf{\sigma} \) so that, in principal axes with \( \mathbf{n}, \mathbf{m}, \) and \( \mathbf{k} \) as unit vectors the stress tensor has representation (see Fig. 17)

\[
[\sigma_{ij}] = \begin{bmatrix}
\sigma_1 & 0 & 0 \\
0 & \sigma_2 & 0 \\
0 & 0 & \sigma_3 \\
\end{bmatrix}.
\tag{2}
\]

Then, the stress vector for the plane with the unit normal \( \mathbf{n}^* \) becomes

\[
\mathbf{p}_{n*} = \sigma_1 n_1^* \mathbf{n} + \sigma_2 n_2^* \mathbf{m} + \sigma_3 n_3^* \mathbf{k},
\tag{3}
\]

where \( n_1^*, n_2^*, \) and \( n_3^* \) are components of \( \mathbf{n}^* \) in the principal coordinate system. Also

\[
\sigma_{n*} = \mathbf{p}_{n*} \cdot \mathbf{n}^* = \sigma_1 (n_1^*)^2 + \sigma_2 (n_2^*)^2 + \sigma_3 (n_3^*)^2.
\tag{4}
\]

By using (4) equation (1) becomes

\[
\sigma_i^2 = \sigma_1 (n_1^*)^2 + \sigma_2 (n_2^*)^2 + \sigma_3 (n_3^*)^2 - [\sigma_1 (n_1^*)^2 + \sigma_2 (n_2^*)^2 + \sigma_3 (n_3^*)^2]^2 .
\tag{5}
\]

We want to determine the extreme of (5) under the restriction

\[
n_1^{*2} + n_2^{*2} + n_3^{*2} = 1 .
\tag{6}
\]

By using the Lagrange multiplier method, we form a new function

\[
F(n_1^*, n_2^*, n_3^*) = \sigma_i^2 - \lambda [(n_1^*)^2 + (n_2^*)^2 + (n_3^*)^2 - 1],
\tag{7}
\]
and find the minimum of (7). Before we form the necessary conditions for the minimum note that by using (6) we could transform (5) as

$$\sigma_i^2 = \sigma_1^2(n_1^*)^2 + \sigma_2^2(n_2^*)^2 + \sigma_3^2(n_3^*)^2 - \sigma_1^2(n_1^*)^2[1 - (n_2^*)^2 - (n_3^*)^2] - \sigma_2^2(n_2^*)^2[1 - (n_1^*)^2 - (n_2^*)^2] - \sigma_3^2(n_3^*)^2[1 - (n_1^*)^2 - (n_2^*)^2] - 2\sigma_1\sigma_2(n_1^*)^2(n_2^*)^2 - 2\sigma_1\sigma_3(n_1^*)^2(n_3^*)^2 - 2\sigma_2\sigma_3(n_2^*)^2(n_3^*)^2$$

$$- 2\alpha_1\alpha_2(n_1^*)^2(n_2^*)^2 - 2\alpha_1\alpha_3(n_1^*)^2(n_3^*)^2 - 2\alpha_2\alpha_3(n_2^*)^2(n_3^*)^2 - 2\alpha_1\alpha_2(n_1^*)^2(n_2^*)^2 - 2\alpha_1\alpha_3(n_1^*)^2(n_3^*)^2 - 2\alpha_2\alpha_3(n_2^*)^2(n_3^*)^2$$

$$+ (\sigma_1 - \sigma_2)^2(n_1^*)^2(n_2^*)^2 + (\sigma_2 - \sigma_3)^2(n_2^*)^2(n_3^*)^2$$

$$+ (\sigma_3 - \sigma_1)^2(n_3^*)^2(n_1^*)^2.$$ \(8\)

By substituting (8) into (7) and differentiating with respect to \(n_i^*, i = 1, 2, 3\) leads to

$$\frac{\partial F}{\partial n_1^*} = 2n_1^*[(\sigma_1 - \sigma_2)^2(n_2^*)^2 + (\sigma_3 - \sigma_1)^2(n_3^*)^2 - \lambda] = 0;$$

$$\frac{\partial F}{\partial n_2^*} = 2n_2^*[(\sigma_1 - \sigma_2)^2(n_1^*)^2 + (\sigma_2 - \sigma_3)^2(n_3^*)^2 - \lambda] = 0;$$

$$\frac{\partial F}{\partial n_3^*} = 2n_3^*[(\sigma_2 - \sigma_3)^2(n_2^*)^2 + (\sigma_3 - \sigma_1)^2(n_1^*)^2 - \lambda] = 0.$$ \(9\)

Solving (9) we obtain two solutions: The first solution is

i) \(n_1^* = \pm 1; \quad n_2^* = 0; \quad n_3^* = 0; \quad \lambda = 0;\)

ii) \(n_1^* = 0; \quad n_2^* = \pm 1; \quad n_3^* = 0; \quad \lambda = 0;\)

iii) \(n_1^* = 0; \quad n_2^* = 0; \quad n_3^* = \pm 1; \quad \lambda = 0.)\) \(10\)

From (10) and (5) it follows that

$$\sigma_i^2 = 0.$$ \(11\)

The solution (10) gives the minimal value \(\sigma_i^2 = 0\) that corresponds to the shear stress in principal planes.

The second solution is

i) \(n_1^* = 0; \quad n_2^* = n_3^* = \pm \frac{\sqrt{2}}{2}; \quad \lambda = \frac{1}{2}(\sigma_2 - \sigma_3)^2;\)

ii) \(n_2^* = 0; \quad n_1^* = n_3^* = \pm \frac{\sqrt{2}}{2}; \quad \lambda = \frac{1}{2}(\sigma_1 - \sigma_3)^2;\)

iii) \(n_3^* = 0; \quad n_1^* = n_2^* = \pm \frac{\sqrt{2}}{2}; \quad \lambda = \frac{1}{2}(\sigma_1 - \sigma_2)^2;\) \(12\)

By substituting (12) into (5) we obtain the following three values of \(\sigma_i\).

These values are candidates for the minimum and maximum of shear stresses

$$\sigma_{t1} = \frac{1}{2}|\sigma_2 - \sigma_3|; \quad \sigma_{t2} = \frac{1}{2}|\sigma_3 - \sigma_1|; \quad \sigma_{t3} = \frac{1}{2}|\sigma_1 - \sigma_2|.$$ \(13\)
From (13) we can get the minimal and maximal values of shear stresses, called the principal shear stresses. Also from (12) and (13) we conclude that the maximum shear stress acts on the plane whose normal bisects the angle between the principal planes of maximum and minimum principal stress, and has a magnitude equal to one-half of the difference between these two principal stresses. Note that in planes where shear stresses have extreme values the normal stresses are not equal to zero. The orientation of planes with maximal and minimal shear stresses is shown in Fig. 18.

1.10 Spherical and deviatoric part of stress tensor

The octahedral plane is a plane whose normal has equal angles with the principal axes at a given point of a body. There are eight octahedral planes, each of which cuts across the corners of the principal element (an element of the body whose faces are parallel to principal planes). One octahedral plane is shown in Fig. 19. The unit normal on the octahedral plane is given as

$$ \mathbf{b} = \pm \frac{1}{\sqrt{3}} \mathbf{n} \pm \frac{1}{\sqrt{3}} \mathbf{m} + \frac{1}{\sqrt{3}} \mathbf{k} . \quad (1) $$

By using (1) and (1.9-2) in (1.7-2) we obtain

$$ \sigma_{oc} = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) = \frac{1}{3} \hat{F}_1 . \quad (2) $$

Similarly, from the expression

$$ \sigma_{toct} = \sqrt{|p_b|^2 - \sigma_{oc}^2} , \quad (3) $$

and (2), we obtain the value of shear stress in the octahedral plane as

$$ \sigma_{toct} = \frac{1}{3} \sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2} $$

$$ = \frac{1}{3} \sqrt{(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{11} - \sigma_{33})^2 + 6(\sigma_{12}^2 + \sigma_{13}^2 + \sigma_{23}^2)} $$

$$ = \frac{1}{3} \sqrt{2\tilde{F}_1^2 - 6\tilde{F}_2} . \quad (4) $$

The last expressions could be simplified if we use principal shear stresses given by (1.9-13). The result is

$$ \sigma_{toct} = \frac{2}{3} \sqrt{\sigma_{t1}^2 + \sigma_{t2}^2 + \sigma_{t3}^2} . \quad (5) $$
The octahedral stresses are important in fracture mechanics. Namely, in the octahedral shear stress theory of elastic failure it is assumed that the failure in elastic material takes place when the octahedral shear stress \( \sigma_{oct} \) reaches the critical value. This theory of elastic failure is known as the Maxwell-von Mises distortion energy theory. The octahedral stress \( \sigma_{oct} \) is sometimes called the hydrostatic or volumetric stress.

The stress tensor \( \sigma \) could be decomposed in two parts as

\[
\sigma_{ij} = \hat{\sigma}_{ij} + \sigma_{oct}\delta_{ij}.
\]  
\[ (6) \]

In equation (6) the part

\[
\hat{\sigma}_{ij} = \sigma_{oct}\delta_{ij},
\]  
\[ (7) \]
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is called the spherical part, and

\[ \hat{\sigma}_{ij} = \sigma_{ij} - \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33})\delta_{ij} , \]  

(8)

is called the deviatoric part of the stress tensor. In writing (8) we used (1.8-2). Note also that the first invariant of the deviatoric part of the stress tensor is equal to zero; that is,

\[ \hat{\sigma}_{ii} = \hat{\sigma}_{11} + \hat{\sigma}_{22} + \hat{\sigma}_{33} = 0 . \]  

(9)

Finally, from (1.8-2) and (7) we obtain for the spherical part of \( \sigma \),

\[ \hat{S}_{ij} = \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33})\delta_{ij} . \]  

(10)

The deviatoric part of \( \sigma \) follows form (6) and (10) as

\[ \hat{\sigma} = \begin{bmatrix} \sigma_{11} - \sigma_{oct} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} - \sigma_{oct} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} - \sigma_{oct} \end{bmatrix} . \]  

(11)

When the stress tensor is referred to the principal axes, expression (11) becomes

\[ \hat{\sigma} = \begin{bmatrix} \sigma_{1} - \sigma_{oct} & 0 & 0 \\ 0 & \sigma_{2} - \sigma_{oct} & 0 \\ 0 & 0 & \sigma_{3} - \sigma_{oct} \end{bmatrix} . \]  

(12)

1.11 Mohr’s stress circles

The state of stress at a point may be represented graphically via Mohr’s circles. We consider, for simplicity, the case when the stress tensor is given in principal axes. Then we have

\[ \sigma = \begin{bmatrix} \sigma_{1} & 0 & 0 \\ 0 & \sigma_{2} & 0 \\ 0 & 0 & \sigma_{3} \end{bmatrix} . \]  

(1)

Suppose that the principal stresses are labeled so that \( \sigma_{1} > \sigma_{2} > \sigma_{3} \). For a plane with the unit normal \( \mathbf{n} \) the stress vector \( \mathbf{p}_{n} \) has the magnitude

\[ |\mathbf{p}_{n}|^2 = |\sigma\mathbf{n}|^2 = \sigma_{1}^2n_{1}^2 + \sigma_{2}^2n_{2}^2 + \sigma_{3}^2n_{3}^2 . \]  

(2)

The normal stress is \( \sigma_{n} = \mathbf{p}_{n} \cdot \mathbf{n} \), so that

\[ \sigma_{n} = \sigma_{1}n_{1}^2 + \sigma_{2}n_{2}^2 + \sigma_{3}n_{3}^2 . \]  

(3)
Since \( \mathbf{n} \) is the unit vector, we also have

\[
\mathbf{n}_1^2 + \mathbf{n}_2^2 + \mathbf{n}_3^2 = 1.
\]  

(4)

From the system (2), (3), (4) we can determine \( \mathbf{n}_1^2 \), \( \mathbf{n}_2^2 \), and \( \mathbf{n}_3^2 \). If we multiply (3) by \( -(\sigma_1 + \sigma_2) \), and (4) by \( \sigma_1 \sigma_2 \) and the results add to (2), we obtain (with \( |\mathbf{p}_{\mathbf{n}}|^2 = \sigma_n^2 + \sigma_t^2 \)),

\[
\sigma_n^2 + \sigma_t^2 - \sigma_n(\sigma_1 + \sigma_2) + \sigma_1 \sigma_2 = \mathbf{n}_3^2[\sigma_3^2 - \sigma_3(\sigma_1 + \sigma_2) + \sigma_1 \sigma_2].
\]  

(5)

From (5) it follows that

\[
\sigma_n^2 - \sigma_n(\sigma_1 + \sigma_2) + \sigma_t^2 + \sigma_1 \sigma_2 = \mathbf{n}_3^2[(\sigma_3 - \sigma_1)(\sigma_3 - \sigma_2)],
\]  

(6)

or

\[
\mathbf{n}_3^2 = \frac{[\sigma_n - \frac{1}{2}(\sigma_1 + \sigma_2)]^2 + \sigma_t^2 - \frac{1}{2}(\sigma_1 - \sigma_2)]^2}{(\sigma_3 - \sigma_1)(\sigma_3 - \sigma_2)}.
\]  

(7)

Similarly, we obtain two more equations of the type (7); that is,

\[
\mathbf{n}_1^2 = \frac{[\sigma_n - \frac{1}{2}(\sigma_2 + \sigma_3)]^2 + \sigma_t^2 - \frac{1}{2}(\sigma_2 - \sigma_3)]^2}{(\sigma_1 - \sigma_3)(\sigma_1 - \sigma_2)};
\]

\[
\mathbf{n}_2^2 = \frac{[\sigma_n - \frac{1}{2}(\sigma_1 + \sigma_3)]^2 + \sigma_t^2 - \frac{1}{2}(\sigma_1 - \sigma_3)]^2}{(\sigma_2 - \sigma_1)(\sigma_2 - \sigma_3)}.
\]  

(8)

From (7) and (8), by using the fact that \( \mathbf{n}_1^2 > 0 \), \( \mathbf{n}_2^2 > 0 \), \( \mathbf{n}_3^2 > 0 \) and the assumption \( \sigma_1 > \sigma_2 > \sigma_3 \), it follows that

\[
[\sigma_n - \frac{1}{2}(\sigma_1 + \sigma_2)]^2 + \sigma_t^2 - \frac{1}{2}(\sigma_1 - \sigma_2)]^2 \geq 0;
\]

\[
[\sigma_n - \frac{1}{2}(\sigma_2 + \sigma_3)]^2 + \sigma_t^2 - \frac{1}{2}(\sigma_2 - \sigma_3)]^2 \geq 0;
\]

\[
[\sigma_n - \frac{1}{2}(\sigma_1 + \sigma_3)]^2 + \sigma_t^2 - \frac{1}{2}(\sigma_1 - \sigma_3)]^2 \leq 0.
\]  

(9)

The region defined by (9) is shown in the \( \sigma_n - \sigma_t \) plane in Fig. 20. From (9) we conclude that the admissible values for \( \sigma_n \) and \( \sigma_t \) must lie in the crescent-shaped shaded region in Fig. 20. The boundary of this region is bounded by three circles called the Mohr's stress circles. Note that Mohr's circle technique gives the total shear stress for a plane with unit normal \( \mathbf{n}^* \).

One may ask for the components of shear stress for the plane with normal \( \mathbf{n}^* \) referred to principal axes. To answer this question we use (1.2-7) to obtain the expression \( \mathbf{p}_{\mathbf{n}} = \sigma_n + \sigma_t \). Then

\[
\sigma_t = \mathbf{p}_{\mathbf{n}^*} - \sigma_{\mathbf{n}^*} = \sigma \mathbf{n}^* - [(\sigma \mathbf{n}^*) \cdot \mathbf{n}^*] \mathbf{n}^*.
\]  

(10)
From (10) and (1), in the coordinate system with principal axes, we have

$$\sigma_t = (\sigma_t)_1 \mathbf{n} + (\sigma_t)_2 \mathbf{m} + (\sigma_t)_3 \mathbf{k}.$$  \hfill (11)

By using (1), (10), (1.9-3), and (1.9-4) the components of the shear stress $\sigma_t$ in the principal coordinate system are

$$\begin{align*}
(\sigma_t)_1 &= (\sigma_1 - \sigma_1(n_1^*)^2)n_1^*; \\
(\sigma_t)_2 &= (\sigma_2 - \sigma_2(n_2^*)^2)n_2^*; \\
(\sigma_t)_3 &= (\sigma_3 - \sigma_3(n_3^*)^2)n_3^*. \\
\end{align*}$$ \hfill (12)

The graphical determination of shear stress components (12) by a procedure similar to the Mohr’s circle could also be constructed (see Almusallam and Thaer (1995)).

### 1.12 Plane state of stress

As a special case of a general state of stress we consider the so-called plane problem. If a body consists of two parallel planes of constant thickness apart and bounded by any closed surface, as shown in Fig. 21, we say that this body is a plane body.

We can associate a number of simplifying assumptions with the plane body to obtain the plane problem of elasticity theory. In order to motivate the mathematical assumptions that we make the following physical assumptions are made.

i) Either no load is applied to the top and bottom surfaces of the body or if there is a load applied to the top and bottom surfaces this load is uniform and produces uniform stress $\sigma_{33} \neq 0$, while $\sigma_{31} = \sigma_{32} = 0$.

ii) Loads on the lateral boundaries are uniformly distributed across the thickness.

iii) Body force components in the $\bar{x}_1$ and $\bar{x}_2$ direction are uniform across the thickness ($f_1 = f_1(x_1, x_2); f_2 = f_2(x_1, x_2)$) and the body force in the $\bar{x}_3$ direction is zero ($f_3 = 0$).
There is no limitation on the thickness of the plane body. The thickness is used to classify the plane problems. Normally a plane state of stress is applied to plane bodies that are relatively thin, compared to other dimensions, whereas plane strain methods are applied to relatively thick plane bodies. Both plane stress and plane strain states are studied in detail in Chapter 6.

On the basis of the assumptions i), ii), and iii) we may say that in the plane problems the stresses do not vary in the $x_3$ direction, and lines parallel to the $x_3$ direction in an undeformed state remain straight and parallel to $x_3$ in deformed state too. Mathematically the plane state of stress is defined as: the stress state at the point is called a plane stress state if the stress vectors at the point are coplanar for all possible planes through this point.

If we orient the coordinate system so that the plane $x_1 - x_2$ is the plane to which stress vectors are parallel (see Fig. 21) then the plane stress state is defined as $(p_1 \times p_2) \cdot p_3 = 0$. On the basis of the preceding definition it follows that

$$\sigma_{33} = \sigma_{31} = \sigma_{32} = 0.$$ (1)

With (1) and the assumptions i), ii), and iii), the equilibrium equations become

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + f_1 = 0; \quad \frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + f_2 = 0.$$ (2)

The stress tensor has the form

$$[\sigma_{ij}] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$ (3)

In writing (3) we assumed that $\sigma_{33} = 0$. If $\sigma_{33} \neq 0$ we can always make $\sigma_{33} = 0$ by superimposing a stress field of different sign than $\sigma_{33}$. From (3)
it follows that one principal value of the stress tensor for plane problems is equal to zero (see Section 1.6).

1.13 Normal and tangential stresses in the plane state of stress

Since the plane problems have numerous applications, we now present a detailed analysis of the stress state at the point for plane problems. Consider an arbitrary point P of the plane body (see Fig. 22). From Fig. 22 it follows that the unit normal for an arbitrary plane is

\[ n = \cos \phi \mathbf{e}_1 + \sin \phi \mathbf{e}_2. \]  

(1)

The stress tensor for plane problems has the form (1.12-3) so that the stress vector becomes

\[
\begin{bmatrix}
(p_n)_1 \\
(p_n)_2 \\
(p_n)_3
\end{bmatrix}
= \begin{bmatrix}
\sigma_{11} & \sigma_{12} & 0 \\
\sigma_{21} & \sigma_{22} & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\cos \phi \\
\sin \phi \\
0
\end{bmatrix},
\]

(2)

or

\[ p_n = (\sigma_{11} \cos \phi + \sigma_{12} \sin \phi) \mathbf{e}_1 + (\sigma_{12} \cos \phi + \sigma_{22} \sin \phi) \mathbf{e}_2. \]  

(3)

The normal and shear stresses are

\[ \sigma_n = p_n \cdot n; \quad \sigma_t = p_n \cdot l, \]

(4)

where \( l \) is the unit vector in the inclined plane; that is,

\[ l = -\sin \phi \mathbf{e}_1 + \cos \phi \mathbf{e}_2. \]  

(5)

By using (1) and (5) in (4) we obtain

\[ \sigma_n = \sigma_{11} \cos^2 \phi + \sigma_{22} \sin^2 \phi + \sigma_{12} \sin 2\phi; \]

\[ \sigma_t = -\frac{1}{2}(\sigma_{11} - \sigma_{22}) \sin 2\phi + \sigma_{12} \cos 2\phi. \]  

(6)
Since $\sigma_n$ and $\sigma_t$ are expressed in terms of one variable $\varphi$, the problem of determining the principal stresses simplifies considerably. Namely, the necessary condition for the existence of an extreme value of $\sigma_n$ becomes

$$\frac{d\sigma_n}{d\varphi} = 2 \left( -\frac{1}{2} (\sigma_{11} - \sigma_{22}) \sin 2\varphi + \sigma_{12} \cos 2\varphi \right) = 0 . \quad (7)$$

From (7) we obtain

$$\tan 2\varphi = \frac{2\sigma_{12}}{\sigma_{11} - \sigma_{22}} , \quad (8)$$

where we assumed that $\sigma_{11} \neq \sigma_{22}$.

If $\sigma_{11} = \sigma_{22}$, from (7) we find

$$\cos 2\varphi = 0 \Rightarrow \varphi_1 = \pi/4; \quad \varphi_2 = 5\pi/4 , \quad (9)$$

that is, the principal planes have angles $\pi/4$ and $5\pi/4$ with the axis $\bar{x}_2$.

When $\sigma_{11} \neq \sigma_{22}$ equation (7) determines two angles, $\varphi_1$ and $\varphi_2 = \varphi_1 + \pi/2$ that define the principal planes. It is easy to see, from (6)_2 and (7) that

$$\frac{d\sigma_n}{d\varphi} |_{\varphi=\varphi_1=\varphi_2 = 2\sigma_t} |_{\varphi=\varphi_1=\varphi_2 = 0} , \quad (10)$$

that is, in the principal planes the shear stresses vanish.

### 1.14 Mohr’s circle for plane state of stress

We determine the value of $\sigma_n$ in the expression (1.13-6)_1 for the angle $\varphi$ defined by (1.13-8). To do this note that from (1.13-8) we have

$$\sin 2\varphi|_{\varphi=\varphi_1} = \frac{\tan 2\varphi}{\sqrt{1 + \tan^2 2\varphi}} = \frac{2\sigma_{12}}{\sqrt{(\sigma_{11} - \sigma_{22})^2 + 4\sigma_{12}^2}} ;$$

$$\cos 2\varphi|_{\varphi=\varphi_1} = \frac{1}{\sqrt{1 + \tan^2 2\varphi}} = \frac{\sigma_{11} - \sigma_{22}}{\sqrt{(\sigma_{11} - \sigma_{22})^2 + 4\sigma_{12}^2}} . \quad (1)$$

By using the trigonometric identities

$$\cos^2 \varphi = \frac{1 + \cos 2\varphi}{2}; \quad \sin^2 \varphi = \frac{1 - \cos 2\varphi}{2} , \quad (2)$$

together with (1), equation (1.13-6)_1 leads to

$$\sigma_{n|\varphi=\varphi_1} = \sigma_1 = \frac{\sigma_{11} + \sigma_{22}}{2} + \frac{1}{2} \sqrt{(\sigma_{11} - \sigma_{22})^2 + 4\sigma_{12}^2} ;$$

$$\sigma_{n|\varphi=\varphi_2} = \sigma_2 = \frac{\sigma_{11} + \sigma_{22}}{2} - \frac{1}{2} \sqrt{(\sigma_{11} - \sigma_{22})^2 + 4\sigma_{12}^2} . \quad (3)$$
Equations (3) give the principal stresses for the plane problems. To obtain Mohr's circle for the plane problem (i.e., the dependence of $\sigma_n$ on $\sigma_t$) we start from (1.13-6). By using (2) it becomes

$$
\sigma_n = \frac{\sigma_{11} + \sigma_{22}}{2} + \frac{\sigma_{11} - \sigma_{22}}{2} \cos 2\varphi + \sigma_{12} \sin 2\varphi ;
$$

$$
\sigma_t = -\frac{\sigma_{11} - \sigma_{22}}{2} \sin 2\varphi + \sigma_{12} \cos 2\varphi .
$$

(4)

From (4) it follows that

$$
\sigma_n - \frac{\sigma_{11} + \sigma_{22}}{2} = \frac{\sigma_{11} - \sigma_{22}}{2} \cos 2\varphi + \sigma_{12} \sin 2\varphi ;
$$

$$
\sigma_t = -\frac{\sigma_{11} - \sigma_{22}}{2} \sin 2\varphi + \sigma_{12} \cos 2\varphi .
$$

(5)

By squaring (5) and (5) and by adding the result, we obtain

$$
\left(\sigma_n - \frac{\sigma_{11} + \sigma_{22}}{2}\right)^2 + \sigma_t^2 = \left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \sigma_{12}^2 .
$$

(6)

Equation (6) connects the normal $\sigma_n$ and shear $\sigma_t$ stress vector components for an arbitrarily inclined plane. In the coordinate system with axes $\sigma_n - \sigma_t$ it represents a circle with the radius $R$ and the center at $(0, a)$. The radius $R$ and the parameter $a$ are determined from

$$
R = \sqrt{\left[\frac{1}{2}(\sigma_{11} - \sigma_{22})\right]^2 + \sigma_{12}^2}; \quad a = \frac{\sigma_{11} + \sigma_{22}}{2} .
$$

(7)

This circle (6) is called Mohr's stress circle and is shown in Fig. 23. Each point on the Mohr's circle represents a state of the stress for a plane with specific $\varphi$. In the special cases when $\varphi = 0$ and $\varphi = 90^\circ$, we have

$$
\varphi = 0; \quad \sigma_n = \sigma_{11}; \quad \sigma_t = \sigma_{12};
$$

$$
\varphi = \frac{\pi}{2}; \quad \sigma_n = \sigma_{22}; \quad \sigma_t = -\sigma_{12} .
$$

(8)

The points corresponding to $\varphi = 0$ and $\varphi = 90^\circ$ are shown in Fig. 23. Suppose that we are given $\sigma_{11}$, $\sigma_{12}$, and $\sigma_{22}$. Then we know the position of the point $B$ ($\varphi = 0$). To construct the Mohr's circle we can use this point and draw a straight line through the point on the $\sigma_n$ axis on the distance $a$ (given by (7)) from the origin. This line defines the radius of the circle. At the points where the circle intersects the $\sigma_n$ axis we obtain the maximal and minimal values of normal stresses (principal stresses). The values $\sigma_1$ and $\sigma_2$ correspond to the planes with orientation given by $\varphi_1$ and $\varphi_2$ satisfying (1.13-8), (see Fig. 23).
Thus, we have
\[ \sigma_1 = a + R, \quad \sigma_2 = a - R, \]
so, by using (7) we obtain
\begin{align*}
\sigma_1 &= \frac{\sigma_{11} + \sigma_{22}}{2} + \frac{1}{2} \sqrt{(\sigma_{11} - \sigma_{22})^2 + 4\sigma_{12}^2}, \\
\sigma_2 &= \frac{\sigma_{11} + \sigma_{22}}{2} - \frac{1}{2} \sqrt{(\sigma_{11} - \sigma_{22})^2 + 4\sigma_{12}^2}.
\end{align*}

in accordance with (3). Equations (4) could be written in simpler form if we introduce angle \( \psi \) as shown in Fig. 24. Namely, \( \psi \) is the angle between the principal plane and an arbitrary plane through the point; that is,
\[ \varphi = \psi + \varphi_1. \]  

By substituting (11) into (4) we obtain
\begin{align*}
\sigma_n &= \frac{\sigma_{11} + \sigma_{22}}{2} \cos[2(\psi + \varphi_1)] + \sigma_{12} \sin[2(\psi + \varphi_1)], \\
\sigma_t &= -\frac{\sigma_{11} + \sigma_{22}}{2} \sin[2(\psi + \varphi_1)] + \sigma_{12} \cos[2(\psi + \varphi_1)].
\end{align*}
Finally when we use (1) in (12) the resulting expression is

\[
\begin{align*}
\sigma_n &= \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 2\psi; \\
\sigma_t &= -\frac{\sigma_1 - \sigma_2}{2} \sin 2\psi.
\end{align*}
\]  

On Mohr’s circle in Fig. 25 we show the points corresponding to \( \varphi = 0 \) and \( \psi = 0 \). For an arbitrary value of \( \psi > 0 \), equation (13) determines the point of the circle (see Fig. 24). If we write (13) as

\[
\sigma_n = a + R \cos 2\psi; \quad \sigma_t = R \sin 2\psi,
\]  

we conclude that: on the Mohr’s circle the angle between the radius that corresponds to the point \( \psi = 0 \) and the radius that corresponds to the point determined by (14) is \( 2\psi \).

Next we determine next the invariants of the stress tensor for the plane state of stress. Since the stress tensor has the form (see (1.12-3))

\[
\sigma_{ij} = \begin{bmatrix}
\sigma_{11} & \sigma_{12} & 0 \\
\sigma_{21} & \sigma_{22} & 0 \\
0 & 0 & 0
\end{bmatrix},
\]  

the invariants (1.8-4) become

\[
\begin{align*}
\hat{I}_1 &= \sigma_{11} + \sigma_{22}; \\
\hat{I}_2 &= \sigma_{11} + \sigma_{22} - \sigma_{12}^2; \\
\hat{I}_3 &= 0.
\end{align*}
\]  

In the principal axes the tensor \([\sigma_{ij}]\) is given as

\[
[\sigma_{ij}] = \begin{bmatrix}
\sigma_1 & 0 & 0 \\
0 & \sigma_2 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]  

From (17) we have

\[
\begin{align*}
\hat{I}_1 &= \sigma_1 + \sigma_2; \\
\hat{I}_2 &= \sigma_1 \sigma_2; \\
\hat{I}_3 &= 0.
\end{align*}
\]  

Finally by comparing (16) and (18) we obtain

\[
\sigma_1 + \sigma_2 = \sigma_{11} + \sigma_{22}; \quad \sigma_1 \sigma_2 = \sigma_{11} \sigma_{22} - \sigma_{12}^2.
\]
1.15 Stresses at the outer surfaces of a body

If the body is loaded by surface forces, then on the outer surface of the body, we have

\[ p_n = \sigma n = \hat{p} \quad \text{on} \quad S_p , \quad (1) \]

where \( S_p \) is the part of the outer surface of the body and \( \hat{p} \) is given. If

\[ \hat{p} = 0 , \quad (2) \]

then we call the surface \( S_p \) the free surface. In Fig. 26a we show part of the body with free surface. Introducing the coordinate system, as shown in Fig. 26b, we obtain

\[ p_3 = 0 . \quad (3) \]

Figure 26

It follows from (3) that the stress tensor, in the coordinate system shown in Fig. 26b, for the point on the free surface, has the representation

\[ \begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} . \quad (4) \]

If we want to determine normal \( \sigma_n \) and shear \( \sigma_t \) stress in a plane that contains \( e_3 \) and has unit normal \( n \), we obtain

\[ p_n = \sigma n , \quad (5) \]

where

\[ n = \cos \varphi e_1 + \sin \varphi e_2 , \quad (6) \]

and \( \varphi \) is shown in Fig. 22. By using (6) in (5) we have

\[ p_n = \left( \sigma_{11} \cos \varphi + \sigma_{12} \sin \varphi \right) e_1 + \left( \sigma_{12} \cos \varphi + \sigma_{22} \sin \varphi \right) e_2 . \quad (7) \]

Equation (7) is identical to (1.13-3). Therefore all conclusions drawn from (1.13-3) apply here too. This is important because most measurements are performed on free surfaces of the body. Therefore for analysis of these measurements we can use the results of Section 1.13.
1.16 Linear state of stress

One special case of the general state of stress at the point is the linear state of stress. It is defined as: the state of stress at a given point of a body is called the linear state of stress if stress vectors for all planes through the point are collinear. In other words the state of stress is linear if

\[ \mathbf{p}_i \times \mathbf{p}_j = 0, \quad i \neq j, \quad i, j = 1, 2, 3 \] (1)

holds at a given point. From the definition it follows that we may select a coordinate system such that in the case of a linear stress state, the stress tensor has the form

\[
[\sigma_{ij}] = \begin{bmatrix}
\sigma & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\] (2)

An example of a linear stress state is the axially loaded straight, prismatic rod shown in Fig. 2. The \( \bar{x}_1 \) axis coincides with the geometrical axis of the rod. For the plane \( A^* \) the unit normal is \( \mathbf{n}^* \) given as

\[ \mathbf{n}^* = \cos \varphi \mathbf{e}_1 + \sin \varphi \mathbf{e}_2, \] (3)

so that

\[ \mathbf{p}^*_n = \sigma \cos \varphi \mathbf{e}_1. \] (4)

By using (4) the normal and shear stress components on the inclined plane \( A^* \) become

\[ \sigma^*_n = \sigma \cos^2 \varphi; \quad \sigma^*_s = -\frac{1}{2} \sigma \sin 2\varphi. \] (5)

Problems

1. The state of stress at the points P, Q, and S are given by stress tensors

\[
[\sigma_{ij}] = \begin{bmatrix}
100 & 20\sqrt{3} & 0 \\
20\sqrt{3} & 60 & 0 \\
0 & 0 & 10
\end{bmatrix}; \quad [\sigma_{ij}] = \begin{bmatrix}
3 & 2 & 2 \\
2 & 2 & 0 \\
2 & 0 & 4
\end{bmatrix};
\]

\[
[\sigma_{ij}] = \begin{bmatrix}
3 & 2 & 4 \\
2 & 0 & 2 \\
4 & 2 & 3
\end{bmatrix},
\]

respectively. The stress components are expressed in [N/mm²]. Determine the principal values and principal directions.

2. Determine the stress vector and normal and shear components of the stress vector if the stress state at the point is given by tensor \( \sigma \) and the plane is defined by the equation written on the right of the stress tensor (the stress components are expressed in N/mm²).
Problems 45

a) \[ [\sigma_{ij}] = \begin{bmatrix} 4 & -4 & 0 \\ -4 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \] plane: \(2x_1 + 3x_2 + \sqrt{3}x_3 = 0;\)

b) \[ [\sigma_{ij}] = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix} \] plane: \(x_1 - 2x_2 + x_3 - 2 = 0;\)

c) \[ [\sigma_{ij}] = \begin{bmatrix} 1 & -3 & \sqrt{2} \\ -3 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 4 \end{bmatrix} \] plane: \(2x_1 - 2x_2 + x_3 - 3 = 0.\)

3. The principal stresses at a point are \(\sigma_1 = 300 \text{ N/mm}^2; \sigma_2 = 124 \text{ N/mm}^2;\) and \(\sigma_3 = 56 \text{ N/mm}^2.\) Determine the largest shear stress and octahedral stress.

4. Show that the octahedral shear stress \(\sigma_{toct}\) given by (1.10-4) is always smaller than the largest shear stress given by (1.9-13).

5. Sketch the Mohr’s circles for the case when \(\sigma_1 = \sigma_2 \neq \sigma_3\) and \(\sigma_1 = \sigma_2 = \sigma_3.\)

6. The stress components in a body are given as functions of coordinates in the form

\[
\begin{align*}
\sigma_{11} &= ax_1 + bx_2^2 + cx_3^3; & \sigma_{12} &= l + mx_3; \\
\sigma_{22} &= dx_1 + ex_2^2 + fx_3^3; & \sigma_{23} &= nx_2 + px_3; \\
\sigma_{33} &= gx_1 + hx_2^2 + kx_3^3; & \sigma_{31} &= qx_1^2 + sx_3^2,
\end{align*}
\]

where \(a, b, ..., s\) are constants. If the body is in equilibrium, determine the corresponding body forces.

7. The solution of the cubic equation \(x^3 + ax^2 + bx + c = 0\) could be obtained as follows.

Use the substitution \(x = y - a/3\) and the transform equation in the form

\[
y^3 + py + q = 0, \quad (a)
\]

where \(p = b - a^2/3\) and \(q = c - ab/3 + 2a^3/27.\)

Substitute \(y = rz,\) to obtain

\[
z^3 + \frac{p}{r^2}z + \frac{q}{r^3} = 0. \quad (b)
\]

Use the substitution \(z = \cos \theta\) and the following trigonometric identity \(\cos 3\theta = 4\cos^3 \theta - 3\cos \theta\) to obtain

\[
z^3 - \frac{3}{4}z - \frac{1}{4}\cos 3\theta = 0. \quad (c)
\]
By comparing (b) and (c) obtain

\[ r = \sqrt{-\frac{4p}{3}}; \quad \cos 3\theta = -\frac{4q}{r^3}. \]  

From (d) determine \( r \) and then from (d)\(_2\) three values of the angle \( \theta \): \( \theta_1 = \theta_0, \) \( \theta_2 = \theta_0 + 2\pi/3, \) and \( \theta_3 = \theta_0 + 4\pi/3. \) The solutions of equation (b) then become

\[ z_1 = \cos \theta_0; \quad z_2 = \cos \theta_2; \quad z_3 = \cos \theta_3. \]  

Use the foregoing procedure to formulate the solution method for characteristic equation (1.6-20),

\[ -\lambda^3 + \hat{I}_1 \lambda^2 - \hat{I}_2 \lambda + \hat{I}_3 = 0. \]  

8. At a given point of a body the principal stresses and principal directions are

a) \( \sigma_1 = 5; \quad \sigma_2 = 4; \quad \sigma_3 = 1; \)

\( n = \frac{1}{2}e_1 - \frac{\sqrt{3}}{2}e_2; \quad m = e_3; \quad k = n \times m; \)

b) \( \sigma_1 = 3; \quad \sigma_2 = 2; \quad \sigma_3 = -1; \)

\( n = \frac{\sqrt{2}}{2}(e_1 + e_3); \quad m = \frac{\sqrt{3}}{3}(-e_1 + e_2 + e_3); \quad k = n \times m; \)

c) \( \sigma_1 = 4; \quad \sigma_2 = 2; \quad \sigma_3 = 1; \)

\( n = \frac{\sqrt{2}}{2}(e_2 - e_3); \quad m = \frac{\sqrt{2}}{2}(e_2 + e_3); \quad k = n \times m. \)

For each case determine the stress tensor in the coordinate system with unit vectors \( e_1, e_2, \) and \( e_3. \)
Chapter 2
Analysis of Strain

2.1 Introduction

The forces that act on a solid (in our case elastic) body produce deformation of the body. In fluids they produce flow. One of the basic problems in continuum mechanics is to describe, quantitatively, the deformation that the body experiences. This is achieved by introduction of measures of deformations. Measures of deformation are based on geometrical quantities that describe deformation of the body. By assumption the body that we treat is a continuum. This implies that the changes in the body are continuous so that a small neighborhood of a given point in an undeformed state remains the neighborhood of the same point in the deformed state too. In other words, no finite (however small) part of the body could be deformed so that its volume becomes equal to zero.

When the body is in motion, in general, its points experience displacement; that is, the position vector of an arbitrary particle changes, with respect to a fixed coordinate, system. One way to describe, quantitatively, the deformation at a given point, say P, is to determine relative changes of length of linear elements originating at P as well as changes in angles between any pair of linear elements originating at P. In differential geometry it is shown that the length of linear elements as well as angles between them could be determined if the metric is known at a given point. Thus, to define measures of deformation we have to calculate the metric at P in the undeformed and deformed states and then compare those two metrics.

In order to be able to calculate metrics, we introduce the mapping that maps the body in the three-dimensional Euclidean space E³. Thus, in the undeformed state the body B occupies a part κ₀ of E³ whereas in the deformed state it occupies the part κ. We call κ₀ and κ the undeformed and deformed configuration of the body. We introduce two coordinate systems, one in κ₀ and one in κ at the same point P where we want to analyze deformation. Let \( \xi_i, i = 1, 2, 3 \) be coordinates in κ₀ and \( \hat{\xi}_i, i = 1, 2, 3 \) coordinates in κ. The coordinate systems, in general, may not be equal (each may be Cartesian or curvilinear coordinate systems). It is assumed that the coordinates \( \xi_i, i = 1, 2, 3 \) of a point in κ are continuous and differentiable functions of the coordinates \( \hat{\xi}_i, i = 1, 2, 3 \) in κ₀. This is specified as the requirement that there exist unique, inevitable relations

\[
\xi_i = f_i(\xi_1, \xi_2, \xi_3); \quad \hat{\xi}_i = g_i(\xi_1, \xi_2, \xi_3) \quad i = 1, 2, 3. \quad (1)
\]

The functions \( f_i \) and \( g_i \) are assumed to be continuous and have continuous
first derivatives with respect to all arguments. The condition that \( f_i \) can be inverted and solved for \( \xi_i \) is satisfied if and only if the Jacobian of the transformation does not vanish. For the study of deformation we can use either coordinates \( \xi_i \) or \( \xi_i \).

In what follows we use the single Cartesian coordinate system with the axes \( \bar{x}_i, \ i = 1, 2, 3 \) and unit vectors \( e_i, \ i = 1, 2, 3 \) in both configurations \( \kappa_0 \) and \( \kappa \). In the first approach, called Lagrange’s representation, the material particle is identified on the basis of its initial coordinates \( X_i (X_i = \xi_i) \) in \( \kappa_0 \). Thus, the coordinates of an arbitrary material point in the deformed configuration \( \kappa (x_i = \xi_i) \) are given as

\[
x_i = x_i(X_1, X_2, X_3) = x_i(X_j) .
\] (2)

The description (2) implies that “the observer is positioned on the fixed particle.” In Eulerian representation “the observer is positioned at the fixed point in space,” that is, we write

\[
X_i = X_i(x_1, x_2, x_3) = X_i(x_j).
\] (3)

The system (2) could be obtained by solving (3) for \( x_i \). Our assumption on \( x_i(X_j) \) guarantees that this inversion is always possible. Both Lagrange’s and Euler’s representations have advantages and disadvantages. Generally speaking in elasticity theory Lagrange’s representation is used and in fluid mechanics the Euler description proves more convenient.

### 2.2 Measures of deformations. Strain tensor

Consider a body \( B \) in two configurations \( \kappa_0 \) and \( \kappa \). We fix two points in \( \kappa_0 \) that we denote by \( P \) and \( Q \). We use the Lagrange representation so that the coordinates \( X_i \) of a point in \( \kappa_0 \) are independent variables. In the deformed configuration \( \kappa \) the point \( P \) is displaced to \( \bar{P} \). The displacement vector of the point \( P \) is defined as

\[
u = r - R ,
\] (1)

where \( R \) is the position vector of \( P \) in \( \kappa_0 \) in the coordinate system \( \bar{x}_i, \ i = 1, 2, 3 \) and \( r \) is the position vector of the same point, with respect to the same coordinate system, in configuration \( \kappa \) (see Fig. 1). Let \( Q \) be a point in the neighborhood of \( P \) in \( \kappa_0 \). In \( \kappa \) this point moves to \( \bar{Q} \). The curve \( C_0 \) connecting \( P \) and \( Q \) in \( \kappa_0 \) changes and becomes \( \bar{C} \) in \( \kappa \) (see Fig. 2). We say that we know the deformation at the point \( P \) if we know:

1) the length in the deformed state \( ds \) of an arbitrary line element originating from \( P \) (such as \( C_0 \)) whose initial length was \( dS \); and

---

\(^1\) In writing (2) we do not distinguish between a function and its value. Otherwise, we would have to write \( x_i = \bar{x}_i(X_j) \), where \( \bar{x}_j(X_j) \) are continuously differentiable functions.
ii) the differences between the angles $\angle(Q, P, R)$ formed by an arbitrary pair of linear elements originating at point $P$ and the angle $\angle(\bar{Q}, \bar{P}, \bar{R})$ formed by the corresponding elements originating at the same point in the deformed state.

We show how both i) and ii) could be obtained. First we determine $dS$. If $R$ is a position vector of $P$ then $R + dR$ is a position vector of $Q$. Suppose, for the moment that we are dealing with Lagrange's representation in a general curvilinear coordinate system, so that

$$R = R(\xi_1, \xi_2, \xi_3).$$

(2)

From (2) we obtain

$$dR = \frac{\partial R}{\partial \xi_i} d\xi_i.$$  

(3)
The arc length connecting points P and Q reads

\[(dS)^2 = d\mathbf{R} \cdot d\mathbf{R} = \frac{\partial \mathbf{R}}{\partial \xi_i} \cdot \frac{\partial \mathbf{R}}{\partial \xi_j} d\hat{\xi}_i d\hat{\xi}_j.\]  

(4)

The vectors \(\frac{\partial \mathbf{R}}{\partial \hat{\xi}_i}, i = 1, 2, 3\) are called base vectors in \(\kappa_0\). If \(\hat{\xi}_i = X_i\) with \(X_i\) being a rectangular Cartesian coordinate system, then \(\frac{\partial \mathbf{R}}{\partial \hat{\xi}_i} = e_i\). Equation (4) can be written as

\[(dS)^2 = G_{ij} d\hat{\xi}_i d\hat{\xi}_j.\]  

(5)

The symmetric (since the scalar product is commutative) two-index system \(G_{ij}\) forms a fundamental or metric tensor in \(\kappa_0\). In the case of an orthogonal Cartesian system, \(G_{ij} = \delta_{ij}\). In the deformed configuration \(\kappa\) we have the position vector \(\mathbf{r} = \mathbf{r}(\hat{\xi}_i)\) so that \((ds)^2 = d\mathbf{r} \cdot d\mathbf{r}\). Therefore

\[d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial \hat{\xi}_i} d\hat{\xi}_i,\]  

(6)

or

\[ds^2 = g_{ij} d\hat{\xi}_i d\hat{\xi}_j.\]  

(7)

The system \(g_{ij}\) is also symmetric and forms a fundamental or metric tensor in the deformed configuration \(\kappa\). Note that from (1) it follows that

\[\frac{\partial \mathbf{r}}{\partial \hat{\xi}_i} = \frac{\partial \mathbf{R}}{\partial \xi_i} + \frac{\partial \mathbf{u}}{\partial \xi_i}.\]  

(8)

As a measure of deformation we take the following quantity

\[(ds)^2 - (dS)^2.\]  

(9)

From (5) and (7) we obtain

\[(ds)^2 - (dS)^2 = [g_{ij} - G_{ij}] d\hat{\xi}_i d\hat{\xi}_j = 2\tilde{E}_{ij} d\hat{\xi}_i d\hat{\xi}_j,\]  

(10)

where

\[\tilde{E}_{ij} = \frac{1}{2} [g_{ij} - G_{ij}].\]  

(11)

The system \(\tilde{E}_{ij}\) forms a second order tensor called the Lagrange-Green strain tensor. In what follows we show that \(\tilde{E}_{ij}\) is a tensor (satisfies the transformation law for second-order tensors) and that with \(\tilde{E}_{ij}\) known we can satisfactorily answer both requirements i) and ii).

Consider now a rectangular Cartesian coordinate system \(X_i\). Then \(G_{ij} = \delta_{ij}\) and \(g_{ij} = (\partial x_i/\partial X_i)(\partial x_j/\partial X_j)\), so that

\[\tilde{E}_{ij} = \frac{1}{2} \left[ \frac{\partial x_m}{\partial X_i} \frac{\partial x_m}{\partial X_j} - \delta_{ij} \right].\]  

(12)
By using (1) we obtain
\[ dx_i = dX_i + du_i . \] (13)

From (13) it follows that
\[ \frac{\partial x_m}{\partial X_i} = \delta_{mi} + \frac{\partial u_m}{\partial X_i} , \] (14)

and \( g_{ij} = (\delta_{mi} + \partial u_m/\partial X_i)(\delta_{mj} + \partial u_m/\partial X_j) \). Finally by substituting (14) in (12) we get

\[ \bar{E}_{11} = \frac{\partial u_1}{\partial X_1} + \frac{1}{2} \left[ \left( \frac{\partial u_1}{\partial X_1} \right)^2 + \left( \frac{\partial u_2}{\partial X_1} \right)^2 + \left( \frac{\partial u_3}{\partial X_1} \right)^2 \right] ; \\
\bar{E}_{22} = \frac{\partial u_2}{\partial X_2} + \frac{1}{2} \left[ \left( \frac{\partial u_1}{\partial X_2} \right)^2 + \left( \frac{\partial u_2}{\partial X_2} \right)^2 + \left( \frac{\partial u_3}{\partial X_2} \right)^2 \right] ; \\
\bar{E}_{33} = \frac{\partial u_3}{\partial X_3} + \frac{1}{2} \left[ \left( \frac{\partial u_1}{\partial X_3} \right)^2 + \left( \frac{\partial u_2}{\partial X_3} \right)^2 + \left( \frac{\partial u_3}{\partial X_3} \right)^2 \right] ; \\
\bar{E}_{12} = \frac{1}{2} \left( \frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1} \right) + \frac{1}{2} \left[ \frac{\partial u_1}{\partial X_1} \frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1} \frac{\partial u_2}{\partial X_2} + \frac{\partial u_3}{\partial X_1} \frac{\partial u_3}{\partial X_2} \right] ; \\
\bar{E}_{23} = \frac{1}{2} \left( \frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2} \right) + \frac{1}{2} \left[ \frac{\partial u_1}{\partial X_2} \frac{\partial u_1}{\partial X_3} + \frac{\partial u_2}{\partial X_2} \frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2} \frac{\partial u_3}{\partial X_3} \right] ; \\
\bar{E}_{13} = \frac{1}{2} \left( \frac{\partial u_1}{\partial X_3} + \frac{\partial u_3}{\partial X_1} \right) + \frac{1}{2} \left[ \frac{\partial u_1}{\partial X_1} \frac{\partial u_1}{\partial X_3} + \frac{\partial u_2}{\partial X_1} \frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_1} \frac{\partial u_3}{\partial X_3} \right] . \] (15)

The system (15) could be written in compact form as
\[ \bar{E}_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} \right) + \frac{1}{2} \left( \frac{\partial u_m}{\partial X_i} \right) \left( \frac{\partial u_m}{\partial X_j} \right) . \] (16)

From the elements \( \bar{E}_{ij} \) we can form a symmetric matrix \( \bar{E} \) as
\[ \bar{[E]}_{ij} = \begin{bmatrix} \bar{E}_{11} & \bar{E}_{12} & \bar{E}_{13} \\
\bar{E}_{21} & \bar{E}_{22} & \bar{E}_{23} \\
\bar{E}_{31} & \bar{E}_{32} & \bar{E}_{33} \end{bmatrix} . \] (17)

We show next that the matrix \( \bar{[E]}_{ij} \) is a second order tensor.\(^2\) Thus, suppose that a new coordinate system \( x_i^* \) is given such that
\[ r = X_i e_i = X_i^* e_i^* . \] (18)

\(^2\)We prove this because we did not prove that \( G_{ij} \) and \( g_{ij} \) are tensors, for otherwise \( \bar{E}_{ij} \) is a tensor as the difference between two tensors.
The unit vectors of the coordinate systems $\bar{x}_i$ and $x^*_i$ are denoted by $e_i$ and $e^*_i$ and are connected through the relation

$$e_i = c_{ki}e^*_k,$$

where $c_{ki} = \cos \angle(x^*_k, \bar{x}_i)$ (see (1.6-6)). Taking the scalar product of (18) with $e_m$, we obtain

$$X_i \delta_{im} = X^*_i c_{km} \delta_{ik},$$

or

$$X_m = c_{im} X^*_i.$$  \hfill (21)

From (21) it follows that

$$dX_i = c_{ki} dX^*_k.$$  \hfill (22)

Since (9) is a scalar its value does not depend on the coordinate system used. Therefore its value must be equal in both systems $\bar{x}_i$, $i = 1, 2, 3$ and $x^*_i$, $i = 1, 2, 3$. Thus, from (10) and (22) we have

$$\frac{1}{2} [(ds)^2 - (dS)^2] = \bar{E}_{11} dX_1^2 + \ldots + 2\bar{E}_{13} dX_1 dX_3$$

or

$$\bar{E}^*_{11} dX_1^2 + \ldots + 2\bar{E}^*_{13} dX_1 dX^*_3,$$  \hfill (23)

where the quantities $\bar{E}^*_{ij}$ are given as

$$\bar{E}^*_{ij} = \left( \bar{E}_{11} c_{i1} + \bar{E}_{12} c_{i2} + \bar{E}_{13} c_{i3} \right) c_{j1} + \left( \bar{E}_{21} c_{i1} + \bar{E}_{22} c_{i2} + \bar{E}_{23} c_{i3} \right) c_{j2}$$

$$+ \left( \bar{E}_{31} c_{i1} + \bar{E}_{32} c_{i2} + \bar{E}_{33} c_{i3} \right) c_{j3}$$

$$= c_{im} \bar{E}^*_{ma} c_{js}.$$  \hfill (24)

From (24) we conclude (as in Section 1.6) that the matrix (17) in the coordinate system $x^*_i$, $i = 1, 2, 3$, that is,

$$[\bar{E}^*_{ij}] = \left[ \begin{array}{ccc} \bar{E}^*_{11} & \bar{E}^*_{12} & \bar{E}^*_{13} \\ \bar{E}^*_{12} & \bar{E}^*_{22} & \bar{E}^*_{23} \\ \bar{E}^*_{13} & \bar{E}^*_{23} & \bar{E}^*_{33} \end{array} \right],$$  \hfill (25)

could be obtained from equation (24) that in direct notation reads

$$\bar{E}^* = Q \bar{E} Q^T,$$  \hfill (26)

where $Q$ is given by (1.6-11). Equation (26) shows that $\bar{E}$ is a (symmetric) second-order tensor.

We determine next the geometrical meaning of the elements $\bar{E}_{ij}$ of the Lagrange-Green strain tensor (17). To do this we fix the point $Q$ that is in the neighborhood of $P$. Suppose further that the position vector of $Q$ differs in the $X_1$ coordinate only from the position vector of $P$. Thus the point $Q$ in $\kappa_0$ has coordinates

$$X_1 + dX_1, \quad X_2, \quad X_3,$$  \hfill (27)
so that \( dX_1 \neq 0, dX_2 = 0, dX_3 = 0 \). Then, \( dX_1 = dS \) and (23) becomes

\[
\frac{1}{2}[(ds)^2 - (dS)^2] = \bar{E}_{11}dX_1^2.
\] (28)

However, since \( dS = dX_1 \) it follows from (28) that

\[
ds = \sqrt{1 + 2\bar{E}_{11}}dX_1.
\] (29)

We define next the relative elongation (also called extension or loosely strain) by the following expression

\[
e_{PQ} = \frac{ds - dS}{dS}.
\] (30)

By using \( dS = dX_1 \) in (30) we obtain

\[
e_{11} = \sqrt{1 + 2\bar{E}_{11}} - 1.
\] (31)

Similarly by choosing \( dX_1 = dX_3 = 0, dX_2 \neq 0 \) and \( dX_1 = dX_2 = 0, dX_3 \neq 0 \), we obtain

\[
e_{22} = \sqrt{1 + 2\bar{E}_{22}} - 1; \quad e_{33} = \sqrt{1 + 2\bar{E}_{33}} - 1.
\] (32)

From (31) and (32) we conclude that: the diagonal elements of the Lagrange-Green strain tensor determine the relative elongation of the line elements along the coordinate axes through relations (31) and (32).

Consider a special case when the components of the strain tensor are small; that is, \( |\bar{E}_{ij}| << \eta \) with \( \eta \) so small that \( \eta^2 \) could be neglected when compared to \( \eta \). Then, from (31) and (32) it follows that

\[
e_{11} = \bar{E}_{11}; \quad e_{22} = \bar{E}_{22}; \quad e_{33} = \bar{E}_{33}.
\] (33)

Equation (33) is interpreted as: if the elements of the Lagrange-Green strain tensor are small (the quadratic elements could be neglected) then the diagonal elements are relative elongations along coordinate axes.

Consider next the case when the points Q and R are in the neighborhood of P in the directions of the \( \bar{x}_1 \) and \( \bar{x}_2 \) axes, respectively (see Fig. 3). The angle \( \angle(Q,P,R) \) is in this case equal to \( \pi/2 \). The difference in the position vectors between points Q, P, and R in \( \kappa_0 \) is denoted by \( d\mathbf{R}_1 \) and \( d\mathbf{R}_2 \) (see Fig. 3). Since

\[
d\mathbf{R}_1 = dX_1 \mathbf{e}_1, \quad d\mathbf{R}_2 = dX_2 \mathbf{e}_2,
\] (34)

it follows that

\[
d\mathbf{r}_1 = d\mathbf{R}_1 + du^1 = dX_1 \mathbf{e}_1 + \frac{\partial u_i}{\partial X_1}dX_1 \mathbf{e}_i;
\]

\[
d\mathbf{r}_2 = d\mathbf{R}_2 + du^2 = dX_2 \mathbf{e}_2 + \frac{\partial u_i}{\partial X_2}dX_2 \mathbf{e}_i.
\] (35)
From (35) we obtain

\[
\begin{align*}
\text{dr}^1 \cdot \text{dr}^2 &= \left[ \left(1 + \frac{\partial u_1}{\partial X_1}\right) dX_1 e_1 + \frac{\partial u_2}{\partial X_1} dX_1 e_2 + \frac{\partial u_3}{\partial X_1} dX_1 e_3 \right] \\
&\quad \cdot \left[ \frac{\partial u_1}{\partial X_2} dX_2 e_1 + \left(1 + \frac{\partial u_2}{\partial X_2}\right) dX_2 e_2 + \frac{\partial u_3}{\partial X_2} dX_2 e_3 \right] \\
&= 2 \bar{E}_{12} dX_1 dX_2 ,
\end{align*}
\]

(36)

where we used (15). However, from (30), (31), and (32) we have

\[
\begin{align*}
\text{dr}^1 \cdot \text{dr}^2 &= \sqrt{1 + \bar{E}_{11}} \sqrt{1 + \bar{E}_{22}} dX_1 dX_2 \cos \theta .
\end{align*}
\]

(37)

By combining (36) and (37), we finally obtain

\[
\cos \theta = \sin \gamma_{12} = \frac{2 \bar{E}_{12}}{\sqrt{1 + 2 \bar{E}_{11}} \sqrt{1 + 2 \bar{E}_{22}}} .
\]

(38)

The angle \( \gamma_{12} = \pi / 2 - \theta \) is called the shear angle between the axes \( \bar{x}_1 \) and \( \bar{x}_2 \). Similarly we can derive the following relations connecting shear angles with the other components of the strain tensor

\[
\begin{align*}
\sin \gamma_{23} &= \frac{2 \bar{E}_{23}}{\sqrt{1 + 2 \bar{E}_{22}} \sqrt{1 + 2 \bar{E}_{33}}} ; \\
\sin \gamma_{13} &= \frac{2 \bar{E}_{13}}{\sqrt{1 + 2 \bar{E}_{11}} \sqrt{1 + 2 \bar{E}_{33}}} .
\end{align*}
\]

(39)

From (38), (39) it follows that: the components \( E_{12} \), \( E_{13} \), and \( E_{23} \) of the Lagrange-Green strain tensor are related to the shear angles between the coordinate axes.

Note that both (31), (33) and (38), (39) are nonlinear relations between the relative elongation and shear angles. For the case when the components
of the strain tensor are small, that is, $|\tilde{E}_{ij}| << \eta$, with $\eta$ small so that $\eta^2$ could be neglected when compared to $\eta$ equations (38), (39) become

$$\frac{1}{2}\gamma_{12} = \tilde{E}_{12}; \quad \frac{1}{2}\gamma_{13} = \tilde{E}_{13}; \quad \frac{1}{2}\gamma_{23} = \tilde{E}_{23}. \quad (40)$$

In the case of the Euler description, the coordinates of a material point in the deformed state $x_i$ are independent variables. Therefore we have $R = R(x_i)$ and $r = r(x_i)$. Thus, for a Cartesian coordinate system in $\kappa$ and $\kappa_0$ we obtain the following expressions for the length of a line element in the undeformed and deformed states

$$(dS)^2 = \frac{\partial R}{\partial x_i} \cdot \frac{\partial R}{\partial x_j} dx_i dx_j = \frac{\partial X_m}{\partial x_i} \frac{\partial X_m}{\partial x_j} dx_i dx_j;$$

$$\frac{\partial r}{\partial x_i} \cdot \frac{\partial r}{\partial x_j} dx_i dx_j = \delta_{ij} dx_i dx_j. \quad (41)$$

Instead of (10) we now have

$$[(dS)^2 - (dS)^2] = \left[\delta_{ij} - \frac{\partial X_m}{\partial x_i} \frac{\partial X_m}{\partial x_j} \right] dx_i dx_j = 2\tilde{E}_{ij} dx_i dx_j. \quad (42)$$

The quantities $\tilde{E}$ defined by (42), that is,

$$\tilde{E}_{ij} = \frac{1}{2} \left[\delta_{ij} - \frac{\partial X_m}{\partial x_i} \frac{\partial X_m}{\partial x_j} \right], \quad (43)$$

constitute a symmetric second-order tensor called the Euler-Almanasi strain tensor. Since $R = r - u$ (see (1)), we obtain

$$\frac{\partial X_m}{\partial x_i} = \delta_{mi} - \frac{\partial u_m}{\partial x_i}. \quad (44)$$

By using (44) in (43), we finally obtain (compare with (16))

$$\tilde{E}_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{1}{2} \left(\frac{\partial u_m}{\partial x_i} \frac{\partial u_m}{\partial x_j}\right). \quad (45)$$
From (45) the components of the Euler-Almanasi strain tensor become

\[
\begin{align*}
\tilde{E}_{11} &= \frac{\partial u_1}{\partial x_1} - \frac{1}{2} \left[ \left( \frac{\partial u_1}{\partial x_1} \right)^2 + \left( \frac{\partial u_2}{\partial x_1} \right)^2 + \left( \frac{\partial u_3}{\partial x_1} \right)^2 \right]; \\
\tilde{E}_{22} &= \frac{\partial u_2}{\partial x_2} - \frac{1}{2} \left[ \left( \frac{\partial u_1}{\partial x_2} \right)^2 + \left( \frac{\partial u_2}{\partial x_2} \right)^2 + \left( \frac{\partial u_3}{\partial x_2} \right)^2 \right]; \\
\tilde{E}_{33} &= \frac{\partial u_3}{\partial x_3} - \frac{1}{2} \left[ \left( \frac{\partial u_1}{\partial x_3} \right)^2 + \left( \frac{\partial u_2}{\partial x_3} \right)^2 + \left( \frac{\partial u_3}{\partial x_3} \right)^2 \right]; \\
\tilde{E}_{12} &= \frac{1}{2} \left( \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) - \frac{1}{2} \left[ \frac{\partial u_1}{\partial x_1} \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_1} \frac{\partial u_3}{\partial x_2} \right]; \\
\tilde{E}_{23} &= \frac{1}{2} \left( \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} \right) - \frac{1}{2} \left[ \frac{\partial u_1}{\partial x_2} \frac{\partial u_1}{\partial x_3} + \frac{\partial u_2}{\partial x_2} \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} \frac{\partial u_3}{\partial x_3} \right]; \\
\tilde{E}_{23} &= \frac{1}{2} \left( \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} \right) - \frac{1}{2} \left[ \frac{\partial u_1}{\partial x_1} \frac{\partial u_1}{\partial x_3} + \frac{\partial u_2}{\partial x_1} \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_1} \frac{\partial u_3}{\partial x_3} \right].
\end{align*}
\] (46)

Similar analysis to the one presented for the Lagrange-Green strain tensor leads to relations connecting the relative elongation and shear angles with the components \( \tilde{E}_{ij} \). The resulting expressions are

\[
\begin{align*}
e_{11} &= \frac{1}{\sqrt{1 - 2\tilde{E}_{11}}} - 1; &e_{22} &= \frac{1}{\sqrt{1 - 2\tilde{E}_{22}}} - 1; \\
e_{33} &= \frac{1}{\sqrt{1 - 2\tilde{E}_{33}}} - 1;
\end{align*}
\]

\[
\begin{align*}
\sin \gamma_{12} &= \frac{2\tilde{E}_{12}}{\sqrt{1 - 2\tilde{E}_{11}} \sqrt{1 - 2\tilde{E}_{22}}}; & \sin \gamma_{13} &= \frac{2\tilde{E}_{13}}{\sqrt{1 - 2\tilde{E}_{11}} \sqrt{1 - 2\tilde{E}_{33}}}; \\
\sin \gamma_{23} &= \frac{2\tilde{E}_{23}}{\sqrt{1 - 2\tilde{E}_{22}} \sqrt{1 - 2\tilde{E}_{33}}}.\end{align*}
\] (47)

In what follows we are concerned with the small deformations. Thus we assume that the quantities \( \frac{\partial u_i}{\partial X_j} \) are small; that is,

\[
\left| \frac{\partial u_i}{\partial X_j} \right| \ll \eta; \quad \left| \frac{\partial u_i}{\partial x_j} \right| \ll \eta.
\] (48)

with \( \eta^2 \) so small that it could be neglected when compared to \( \eta \). In this case the strain tensor is called the small strain tensor or linear strain tensor.
Its components \( E_{ij} \) become (see (16))

\[
\bar{E}_{ij} \approx \frac{1}{2} \left( \frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} \right) = E_{ij}.
\]

Introducing the quantity \( \omega_{ij} \) by

\[
\omega_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial X_j} - \frac{\partial u_j}{\partial X_i} \right),
\]
the Lagrange-Green strain tensor could be written in terms of a linear strain tensor as

\[
\bar{E}_{ij} = E_{ij} + \frac{1}{2} (E_{ki} + \omega_{ki})(E_{kj} + \omega_{kj}).
\]

The physical significance of the (tensor) quantity \( \omega_{ij} \) is discussed in Section 2.4. In most applications the deformation of the body is so small that it suffices to use the small strain tensor \( E \). Thus, when we say strain tensor we mean the small strain tensor.

The diagonal elements of the strain tensor are relative elongation along the axes \( \hat{x}_i \) (see (33)) and off diagonal elements are one half of the shear angles (see (40)). The question that we address now is could we determine relative elongation for an arbitrary direction and shear angle for two arbitrary line elements originating at \( P \). If so, then the strain tensor describes completely deformation at the point, according to our requirements i) and ii). We treat this question now.

### 2.3 Extension and shear angle for arbitrary directions

From (2.2-10), in the case of Cartesian coordinate system \( \hat{\xi}_i = X_i \), we obtain

\[
\frac{1}{2} \left[ \left( \frac{ds}{dS} \right)^2 - 1 \right] = \bar{E}_{ij} \frac{dX_i}{dS} \frac{dX_j}{dS}.
\]

The relation (1) is valid for any deformation. We assume that the deformation is small so that we replace \( \bar{E}_{ij} \) with \( E_{ij} \). Let \( n \) be an arbitrary unit vector in the direction of a line element originating at the point \( P \) for which we want to find the relative elongation (see Fig. 1). The position vector of \( Q \) with respect to \( P \) is \( d\mathbf{R} = dX_i e_i \). The relative elongation in the direction defined by \( n = n_i e_i \) is

\[
e_n = \frac{ds - dS}{dS}.
\]

Also

\[
\frac{dX_i}{dS} = \cos \angle(d\mathbf{R}, \hat{x}_i) = \frac{d\mathbf{R}}{dS} \cdot e_i = n_i,
\]
so that (1) becomes
\[ e_n + \frac{1}{2} e_n^2 = \bar{E}_{ij} n_i n_j , \] (4)
where we used (2). If we neglect \( |\partial u_i / \partial X_j|^2 \) in the expressions for \( \bar{E}_{ij} \) then the components \( \bar{E}_{ij} \) become \( E_{ij} \). The same order of approximation requires that \( e_n^2 \) be neglected so that (4) becomes
\[ e_n = E_{ij} n_i n_j , \] (5)
giving the relative elongation in an arbitrary direction \( n \). Note the analogy between (5) and (1.7-2). The stress tensor is replaced by the strain tensor and the normal component of the stress vector by the relative elongation. From (5) we conclude that: the strain tensor \( E_{ij} \) determines the relative elongation for an arbitrary direction. Thus the requirement 1) of Section 2.2 is satisfied.

Let \( d\mathbf{R}^1 \) and \( d\mathbf{R}^2 \) be two infinitesimal vectors connecting the points \( Q \) and \( R \) with \( P \) (see Fig. 2) in \( \kappa_0 \). The shear angle for those two line elements originating at \( P \) is

\[ \gamma_{nm} = \angle(d\mathbf{R}^1, d\mathbf{R}^2) - \angle(dr^1, dr^2) = \theta - \theta' , \] (6)

where we used \( dr^1 \) and \( dr^2 \) to denote the vectors connecting \( \bar{Q} \) and \( \bar{R} \) with \( \bar{P} \) in \( \kappa \). Also
\[ d\mathbf{R}^1 \cdot d\mathbf{R}^2 = dS_1 dS_2 \cos \theta , \] (7)
where \( dS_1 = |d\mathbf{R}^1| \) and \( dS_2 = |d\mathbf{R}^2| \). In \( \kappa \) the vectors \( dr^1 \) and \( dr^2 \) are given as

\[ dr^1 = d\mathbf{R}^1 + du^1 = (\delta_{ij} + \frac{\partial u_i}{\partial X_j}) dX_j^1 e_i; \]
\[ dr^2 = d\mathbf{R}^2 + du^2 = (\delta_{ij} + \frac{\partial u_i}{\partial X_j}) dX_j^2 e_i . \] (8)

By definition of the scalar product, we have

\[ dr^1 \cdot dr^2 = |dr^1||dr^2| \cos \theta' = (1 + e_n)(1 + e_m)dS_1 dS_2 \cos \theta' , \] (9)

where we used \( n \) and \( m \) to denote the unit vectors along \( d\mathbf{R}^1 \) and \( d\mathbf{R}^2 \); that is,
\[ n = \frac{d\mathbf{R}^1}{dS_1} = \frac{dX_1^i}{dS_1} e_i; \quad m = \frac{d\mathbf{R}^2}{dS_2} = \frac{dX_2^i}{dS_2} e_i. \] (10)

By using (8), we obtain
\[ dr^1 \cdot dr^2 = d\mathbf{R}^1 \cdot d\mathbf{R}^2 + d\mathbf{R}^1 \cdot du^2 + d\mathbf{R}^2 \cdot du^1 + du^1 \cdot du^2 . \] (11)

Combining (9) and (11) and dividing by \( dS_1 dS_2 \) it follows that
\[ (1 + e_n)(1 + e_m) \cos \theta' - \cos \theta = n_i \frac{\partial u_i}{\partial X_j} m_j + m_i \frac{\partial u_i}{\partial X_j} n_j , \] (12)
2.3 Extension and shear angle for arbitrary directions

where we neglected the last term in (11) since it is of the order $|\partial u_i/\partial X_j|^2$. To see this, note that

$$\frac{d\mathbf{u}^1 \cdot d\mathbf{u}^2}{dS_1 dS_2} = \left( \frac{\partial u_i}{\partial X_j} dX_j^1 e_i \cdot \frac{\partial u_m}{\partial X_r} dX_r^2 e_m \right) \frac{1}{dS_1 dS_2} = \frac{\partial u_i}{\partial X_j} n_j \frac{\partial u_i}{\partial X_r} m_r .$$

(13)

The right-hand side in (12) could be transformed as follows

$$n_i \frac{\partial u_i}{\partial X_j} m_j + m_i \frac{\partial u_i}{\partial X_j} n_j = \frac{1}{2} \left( \frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} \right) n_i m_j$$

$$+ \frac{1}{2} \left( \frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} \right) m_i n_j$$

$$= E_{ij} (n_i m_j + m_i n_j) = 2E_{ij} n_i m_j .$$

(14)

Introducing the shear angle $\gamma_{nm}$ as

$$\theta' = \theta - \gamma_{nm} ,$$

(15)

and by using (14) and (15) in (12), we obtain

$$\gamma_{nm} \sin \theta = 2E_{ij} n_i m_j - (e_n + e_m) \cos \theta .$$

(16)

In writing (16) we assumed that $\gamma_{nm}$ is small and neglected the terms of the order $\gamma_{nm}^2$ since they are of the order $|\partial u_i/\partial X_j|^2$. From (16), in the special case when $\mathbf{n} \perp \mathbf{m}$, ($\theta = \pi/2$) we obtain

$$\frac{1}{2} \gamma_{nm} = E_{ij} n_i m_j .$$

(17)

The result of equation (16), or in the special case (17), is stated as: the strain tensor $E_{ij}$ determines the shear angle for two arbitrary directions originating at an arbitrary point $P$.

We comment on other strain measures. They are often defined with special requirements in mind. For example, some are multiplicatively symmetrical (doubling in length corresponds to double value of the strain measure and halving in length corresponds to halving of the strain measure). If we define $\lambda = ds/dS$ then $\lambda$ is multiplicatively symmetrical. Other strain measures are additively symmetrical (a measure is additively symmetrical if when doubled in length it is equal to the strain measure for halving in length with a negative sign). For example, the following two measures

$$f = \ln \lambda, \quad \text{where} \quad \lambda = ds/dS \quad \text{and} \quad g = -\ln[\tan \frac{1}{2}(\frac{1}{2}\pi - \gamma)] ,$$

where $\gamma$ is the shear angle, are additively symmetrical.

The logarithmic strain measures are often called “the natural strain measures.” Note that relative elongation defined by (2.2-30) is neither additively nor multiplicatively symmetrical. For a review of various strain mea-
Chapter 2 Analysis of Strain

sures used in elasticity theory, see Truesdell and Toupin (1960) and Lurie (1980).

2.4 Infinitesimal rotation

In this section we want to give a physical interpretation of the quantity \( \omega_{ij} \) defined by (2.2-50). Consider point P and the neighboring point Q (see Fig. 1) in \( \kappa_0 \). In \( \kappa \) these points move to \( \bar{P} \) and \( \bar{Q} \), respectively. Suppose that the displacement vector of P is

\[
\mathbf{r} - \mathbf{R} = \mathbf{u}(P),
\]

(1)

or

\[
\mathbf{u} = u_i \mathbf{e}_i.
\]

(2)

The displacement vector of the point Q is

\[
\mathbf{u}(Q) = \mathbf{u}(P) + d\mathbf{u}.
\]

(3)

In general, \( d\mathbf{u}_i \) is given as

\[
d\mathbf{u}_i = \frac{\partial u_i}{\partial X_j} dX_j,
\]

(4)

so that from (4) we obtain

\[
d\mathbf{u}_i = \left[ \frac{1}{2} \left( \frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} \right) + \frac{1}{2} \left( \frac{\partial u_i}{\partial X_j} - \frac{\partial u_j}{\partial X_i} \right) \right] dX_j.
\]

(5)

By introducing the rotation tensor \( \omega \) with the components \( \omega_{ij} \) given by (2.2-50),

\[
\omega_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial X_j} - \frac{\partial u_j}{\partial X_i} \right),
\]

(6)

the relation (5) becomes

\[
d\mathbf{u}_i = [E_{ij} + \omega_{ij}] dX_j,
\]

(7)

where \( E_{ij} \) is the (linear) strain tensor. Note that the tensor \( \omega \) is skew-symmetric; that is, \( \omega = -\omega^T \). We show next that the part of the displacement vector

\[
d\mathbf{u} = \omega_{ij} dX_j \mathbf{e}_i,
\]

(8)

---

3 The Lagrange-Green and Euler-Almanasi strain tensors are not the only tensorial measures of deformation used. A few others are the right Cauchy-Green tensor, Finger's tensor (left Cauchy-Green tensor), Reiner strain tensor, Klichevski's strain tensor, etc. In geometrical interpretation (such as (2.3-31) and (2.3-39)) strain measures lead to different results.
corresponds to the infinitesimal rigid body rotation of the neighborhood of the point $P$. Consider part of the body in the $\bar{x}_1 - \bar{x}_2$ coordinate plane (see Fig. 4).

Figure 4

If the neighborhood of $P$ makes infinitesimal rotation for an angle $w_3$ "as a rigid body," then

$$w_3 = \frac{\partial u_1}{\partial X_2} = \frac{\partial u_2}{\partial X_1},$$

or

$$-w_3 = \frac{1}{2} \left( \frac{\partial u_1}{\partial X_2} + \frac{\partial u_1}{\partial X_2} \right) = \frac{1}{2} \left( \frac{\partial u_1}{\partial X_2} + \left( -\frac{\partial u_2}{\partial X_1} \right) \right) = \omega_{12}. \tag{10}$$

Therefore, one of the components of the tensor $\omega$ is connected with the angle of rotation of the neighborhood as a rigid body. We can connect the other components of $\omega$ with rotation angles about the coordinate axes so that

$$w_1 = -\omega_{23}; \quad w_2 = \omega_{13}; \quad w_3 = -\omega_{12}. \tag{11}$$

Since the tensor $\omega$ is skew-symmetric (see (6)),

$$\omega_{ij} = -\omega_{ji}, \tag{12}$$

it has a matrix representation in the system $\bar{x}_i$ given by

$$[\omega_{ij}] = \begin{bmatrix} 0 & \omega_{12} & \omega_{13} \\ -\omega_{12} & 0 & \omega_{23} \\ -\omega_{13} & -\omega_{23} & 0 \end{bmatrix}. \tag{13}$$

\footnote{One way to show this is to note that for fixed point $P$ the terms $\omega_{ij}$ are constants. Then (8) could be integrated to give $\hat{u} = \omega_{ij} X_j e_i$. Now calculate the infinitesimal strain tensor $E^\omega$ for the displacement field $\hat{u}$. The result is $E^\omega_{ij} = \frac{1}{2} \left[ (\partial u^\omega_i / \partial X_j) + (\partial u^\omega_j / \partial X_i) \right] = 0$. It follows therefore that $\hat{u}$ corresponds to the displacement "as a rigid body" of the neighborhood of point $P$ with respect to point $P$. It is important to stress that the Lagrange-Green strain tensor given by (2.2-16) is not equal to zero for the displacement field $\hat{u} = \omega_{ij} X_j e_i$ (see Problem 11 at the end of this section).}
Let \( \mathbf{w} \) be the vector of infinitesimal rotation, that is, the axial vector of a skew-symmetric tensor \( \mathbf{\omega} \),

\[
\mathbf{w} = w_1 \mathbf{e}_1 + w_2 \mathbf{e}_2 + w_3 \mathbf{e}_3 .
\]

(14)

The vector \( \mathbf{w} \) has components

\[
w_1 = -\omega_{23}; \quad w_2 = \omega_{13}; \quad w_3 = -\omega_{12}.
\]

(15)

Equation (8) can now be written as

\[
d\mathbf{u} = \mathbf{w} \times d\mathbf{X}.
\]

(16)

Expression (16) shows, again, that the second term in (7), that is, \( d\mathbf{u} \), given by (8), represents the “rigid body displacement” of the neighborhood of \( \mathbf{P} \) since the left-hand side of (16) is the well-known Euler formula for the infinitesimal rotation of the rigid body. Note that (6) could be written as

\[
\mathbf{w} = \frac{1}{2} \text{curl } \mathbf{u} = \frac{1}{2} \nabla \times \mathbf{u} = \frac{1}{2} \left( \frac{\partial u_3}{\partial X_2} - \frac{\partial u_2}{\partial X_3} \right) \mathbf{e}_1 + \frac{1}{2} \left( \frac{\partial u_1}{\partial X_3} - \frac{\partial u_3}{\partial X_1} \right) \mathbf{e}_2 + \frac{1}{2} \left( \frac{\partial u_2}{\partial X_1} - \frac{\partial u_1}{\partial X_2} \right) \mathbf{e}_3.
\]

(17)

By using (16) and (7) in (3), we obtain

\[
\mathbf{u}(Q) = \mathbf{u}(P) + \mathbf{E} d\mathbf{X} + \mathbf{w} \times d\mathbf{X}.
\]

(18)

Equation (18) is the mathematical expression of Helmholtz’s theorem: the displacement vector of the point \( \mathbf{Q} \) that is in the neighborhood of \( \mathbf{P} \) consists of three parts: translation \( \mathbf{u}(\mathbf{P}) \); pure deformation \( \mathbf{E} d\mathbf{X} \), and rigid body rotation \( \mathbf{w} \times d\mathbf{X} \).

2.5 Principal directions of strain tensor

Since \( \mathbf{E} \) is a symmetric second-order tensor we can use the analysis presented in Section 1.6 to conclude that there exist three mutually orthogonal directions, principal directions, in which the strain tensor has only diagonal elements. Those directions are called the principal directions of the strain tensor. The corresponding diagonal elements are called the principal strains. Therefore, in the coordinate system whose axes are directed along the principal directions, the strain tensor has the representation

\[
[E_{ij}] = \begin{bmatrix}
  e_1 & 0 & 0 \\
  0 & e_2 & 0 \\
  0 & 0 & e_3 \\
\end{bmatrix}.
\]

(1)
The principal strains $e_i, i = 1, 2, 3$ are determined from the equation

$$-e^3 + \bar{I}_1e^2 - \bar{I}_2e + \bar{I}_3 = 0,$$

(2)

where parameters $\bar{I}_i, i = 1, 2, 3$ are given by the following expressions

$$\bar{I}_1 = E_{11} + E_{22} + E_{33},$$

$$\bar{I}_2 = \text{det} \begin{vmatrix} E_{22} & E_{23} \\ E_{32} & E_{33} \end{vmatrix} + \text{det} \begin{vmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{vmatrix} + \text{det} \begin{vmatrix} E_{11} & E_{13} \\ E_{31} & E_{33} \end{vmatrix},$$

(3)

$$\bar{I}_3 = \text{det} \begin{vmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{vmatrix}.$$  

The principal strains satisfy

$$(e - e_1)(e - e_2)(e - e_3) = 0,$$

(4)

so that, from (2) and (4) we obtain the invariants of the strain tensor as

$$\bar{I}_1 = e_1 + e_2 + e_3; \quad \bar{I}_2 = e_1e_2 + e_2e_3 + e_1e_3; \quad \bar{I}_3 = e_1e_2e_3.$$  

(5)

Note that $e_i, i = 1, 2, 3$ are real, since $\mathbf{E}$ is a symmetric tensor. The principal strains $e_i$ could be equal to zero (if $ds = dS$) or they could have positive or negative sign (depending on whether $ds$ is larger or smaller than $dS$).

The principal directions $\mathbf{n}$, $\mathbf{m}$, and $\mathbf{k}$ are determined from the system of equations that for $\mathbf{n}$ read

$$(E_{11} - e_1)n_1 + E_{12}n_2 + E_{13}n_3 \quad = \quad 0;$$

$$E_{21}n_1 + (E_{22} - e_1)n_2 + E_{23}n_3 \quad = \quad 0;$$

$$E_{31}n_1 + E_{32}n_2 + (E_{33} - e_1)n_3 \quad = \quad 0;$$

$$n_1^2 + n_2^2 + n_3^2 \quad = \quad 1.$$  

(6)

The principal values of $\mathbf{E}$, that is, $e_i, i = 1, 2, 3$, are the extreme values of relative elongation at a given point.

### 2.6 Strain tensor in coordinate systems

The Cartesian coordinates of an arbitrary point, when expressed in terms of cylindrical coordinates $r$, $\theta$, and $z$ are given by the following relations (see Fig. 1-10)

$$x_1 = r \cos \theta; \quad x_2 = r \sin \theta; \quad x_3 = z.$$  

(1)
The displacement vector $\mathbf{u}$ may be written as

$$
\mathbf{u} = u_r \mathbf{e}_r + u_\theta \mathbf{e}_\theta + u_z \mathbf{e}_z,
$$

where $u_r$, $u_\theta$, and $u_z$ are functions of $r$, $\theta$, and $z$. Since $\mathbf{r} = \mathbf{R} + \mathbf{u}$, we have

$$
dr = d\mathbf{R} + d\mathbf{u}.
$$

Also $\mathbf{R} = re_r + ze_z$, so that\(^5\)

$$
d\mathbf{R} = dre_r + rd\theta e_\theta + dze_z
$$

$$
d\mathbf{u} = \frac{\partial \mathbf{u}}{\partial r} dr + \frac{\partial \mathbf{u}}{\partial \theta} d\theta + \frac{\partial \mathbf{u}}{\partial z} dz
$$

$$
= \frac{\partial u_r}{\partial r} dre_r + \left( \frac{\partial u_r}{\partial \theta} e_r + u_r e_\theta \right) d\theta + \frac{\partial u_r}{\partial z} dze_r
$$

$$
+ \frac{\partial u_\theta}{\partial r} dre_\theta + \left( \frac{\partial u_\theta}{\partial \theta} e_\theta + u_\theta (-e_r) \right) d\theta + \frac{\partial u_\theta}{\partial z} dze_\theta
$$

$$
+ \frac{\partial u_z}{\partial r} dze_z + \frac{\partial u_z}{\partial \theta} d\theta e_z + \frac{\partial u_z}{\partial z} dze_z. \tag{4}
$$

Since $(dS)^2 = d\mathbf{R} \cdot d\mathbf{R}$; $(ds)^2 = dr \cdot dr$, by using (3) we obtain

$$
\frac{1}{2}((ds)^2 - (dS)^2) = d\mathbf{R} \cdot d\mathbf{u} + \frac{1}{2}d\mathbf{u} \cdot d\mathbf{u}
$$

$$
= \bar{E}_{rr}(dr)^2 + \bar{E}_{\theta\theta}(r d\theta)^2 + \bar{E}_{zz}(dz)^2
$$

$$
+ 2\bar{E}_{r\theta} r d\theta dr + 2\bar{E}_{\theta z} r d\theta dz + 2\bar{E}_{z r} dz dr. \tag{5}
$$

In (5) the quantities $\bar{E}_{rr}, ..., \bar{E}_{z r}$ are dimensionless; that is, they represent the physical components of the strain tensor. From (4) it follows that

$$
\bar{E}_{rr} = \frac{\partial u_r}{\partial r} + \frac{1}{2} \left[ \left( \frac{\partial u_r}{\partial r} \right)^2 + \left( \frac{\partial u_\theta}{\partial r} \right)^2 + \left( \frac{\partial u_z}{\partial r} \right)^2 \right];
$$

$$
\bar{E}_{\theta\theta} = \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}
$$

$$
+ \frac{1}{2} \left[ \left( \frac{1}{r} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} \right)^2 + \left( \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \right)^2 + \left( \frac{1}{r} \frac{\partial u_z}{\partial \theta} \right)^2 \right];
$$

$$
\bar{E}_{zz} = \frac{\partial u_z}{\partial z} + \frac{1}{2} \left[ \left( \frac{\partial u_r}{\partial z} \right)^2 + \left( \frac{\partial u_\theta}{\partial z} \right)^2 + \left( \frac{\partial u_z}{\partial z} \right)^2 \right];
$$

---

\(^5\)From the equations $\mathbf{e}_r = \cos \theta \mathbf{e}_1 + \sin \theta \mathbf{e}_2$, $\mathbf{e}_\theta = -\sin \theta \mathbf{e}_1 + \cos \theta \mathbf{e}_2$, $\mathbf{e}_z = \mathbf{e}_3$, we can obtain the derivatives of $\mathbf{e}_r$, $\mathbf{e}_\theta$, and $\mathbf{e}_z$ with respect to $r$, $\theta$, and $z$ that are needed in (4).
\[
\tilde{E}_{r\theta} = \frac{1}{2} \left( \frac{1}{r} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} + \frac{\partial u_\theta}{\partial r} \right) + \frac{1}{2} \left\{ \frac{\partial u_r}{\partial r} \left( \frac{1}{r} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} \right) + \frac{\partial u_\theta}{\partial r} \left( \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \right) + \frac{\partial u_z}{\partial r} \frac{1}{r} \frac{\partial u_z}{\partial \theta} \right\}; \\
\tilde{E}_{\theta z} = \frac{1}{2} \left( \frac{\partial u_\theta}{\partial z} + \frac{1}{r} \frac{\partial u_z}{\partial \theta} \right) + \frac{1}{2} \left\{ \frac{\partial u_r}{\partial z} \left( \frac{1}{r} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} \right) + \frac{\partial u_\theta}{\partial z} \left( \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \right) + \frac{\partial u_z}{\partial z} \frac{1}{r} \frac{\partial u_z}{\partial \theta} \right\}; \\
\tilde{E}_{z r} = \frac{1}{2} \left( \frac{\partial u_z}{\partial z} + \frac{1}{\sin \theta} \frac{\partial u_z}{\partial \varphi} \right) + \frac{1}{2} \left\{ \frac{\partial u_r}{\partial \varphi} \frac{\partial u_r}{\partial \varphi} + \frac{\partial u_\varphi}{\partial \varphi} \frac{\partial u_\varphi}{\partial \varphi} + \frac{\partial u_z}{\partial \varphi} \frac{\partial u_z}{\partial \varphi} \right\}. \tag{6}
\]

In a spherical coordinate system with coordinates \(\rho, \theta, \) and \(\varphi\) we have the following connection with Cartesian coordinates (see Fig. 5)

\[
x_1 = \rho \sin \varphi \cos \theta; \quad x_2 = \rho \sin \varphi \sin \theta; \quad x_3 = \rho \cos \varphi. \tag{7}
\]

Figure 5

The unit vectors \(e_\rho, e_\varphi, \) and \(e_\theta\) and their derivatives could be expressed as in (1.5-8). Then \(\mathbf{R} = \rho e_\rho\) so that

\[
d\mathbf{R} = d\rho e_\rho + \rho d\varphi e_\varphi + \rho \sin \varphi d\theta e_\theta; \quad d\mathbf{u} = \frac{\partial \mathbf{u}}{\partial \rho} d\rho + \frac{\partial \mathbf{u}}{\partial \varphi} d\varphi + \frac{\partial \mathbf{u}}{\partial \theta} d\theta, \tag{8}
\]

where

\[
\mathbf{u} = u_\rho e_\rho + u_\theta e_\theta + u_\varphi e_\varphi. \tag{9}
\]

From the definition

\[
\frac{1}{2}((ds)^2 - dS^2) = d\mathbf{R} \cdot d\mathbf{u} + \frac{1}{2} \mathbf{u} \cdot d\mathbf{u} = \tilde{E}_{\rho \rho} (d\rho)^2 + \tilde{E}_{\varphi \varphi} (\rho d\varphi)^2 + \tilde{E}_{\theta \theta} (\rho \sin \varphi d\theta)^2 + 2 \tilde{E}_{\rho \varphi} d\rho \rho d\varphi + 2 \tilde{E}_{\rho \theta} \rho d\varphi \rho \sin \varphi d\theta + 2 \tilde{E}_{\theta \rho} \rho \sin \varphi d\theta d\rho, \tag{10}
\]
we obtain the components of the strain tensor by substituting (8) and (9) in (10)

\[ \tilde{\varepsilon}_{\rho \rho} = \frac{\partial u_{\rho}}{\partial \rho} + \frac{1}{2} \left[ \left( \frac{\partial u_{\rho}}{\partial \rho} \right)^2 + \left( \frac{\partial u_{\varphi}}{\partial \rho} \right)^2 + \left( \frac{\partial u_{\theta}}{\partial \rho} \right)^2 \right] ; \]

\[ \tilde{\varepsilon}_{\varphi \varphi} = \frac{1}{\rho} \frac{\partial u_{\varphi}}{\partial \varphi} + u_{\rho} + \frac{1}{2} \left[ \left( \frac{1}{\rho} \frac{\partial u_{\rho}}{\partial \varphi} - \frac{u_{\varphi}}{\rho} \right)^2 + \left( \frac{\partial u_{\varphi}}{\partial \rho} + \frac{u_{\rho}}{\rho} \right)^2 + \left( \frac{1}{\rho} \frac{\partial u_{\theta}}{\partial \varphi} \right)^2 \right] ; \]

\[ \tilde{\varepsilon}_{\theta \theta} = \frac{1}{\rho \sin \varphi} \frac{\partial u_{\theta}}{\partial \theta} + u_{\rho} + \frac{u_{\varphi}}{\rho} \cot \varphi + \frac{1}{\sin^2 \varphi} \left[ \left( \frac{1}{\rho} \frac{\partial u_{\rho}}{\partial \theta} - \frac{u_{\theta}}{\rho} \sin \varphi \right)^2 + \left( \frac{1}{\rho} \frac{\partial u_{\varphi}}{\partial \theta} + \frac{u_{\varphi}}{\rho} \sin \varphi + \frac{u_{\rho}}{\rho} \cos \varphi \right)^2 \right] ; \]

\[ \tilde{\varepsilon}_{\rho \varphi} = \frac{1}{2} \left( \frac{1}{\rho} \frac{\partial u_{\rho}}{\partial \varphi} - \frac{u_{\varphi}}{\rho} + \frac{u_{\rho}}{\rho} \right) + \frac{1}{2} \left[ \frac{\partial u_{\varphi}}{\partial \rho} \left( \frac{1}{\rho} \frac{\partial u_{\rho}}{\partial \varphi} - \frac{u_{\varphi}}{\rho} \right) \right] + \frac{\partial u_{\varphi}}{\partial \rho} \left( \frac{1}{\rho} \frac{\partial u_{\rho}}{\partial \varphi} + \frac{u_{\rho}}{\rho} \right) + \frac{\partial u_{\theta}}{\partial \rho} \frac{\partial u_{\varphi}}{\partial \varphi} ; \]

\[ \tilde{\varepsilon}_{\varphi \theta} = \frac{1}{2} \left( \frac{1}{\rho \sin \varphi} \frac{\partial u_{\varphi}}{\partial \theta} - \cot \varphi \frac{u_{\theta}}{\rho} + \frac{1}{\rho} \frac{\partial u_{\theta}}{\partial \varphi} \right) + \frac{1}{2} \left( \frac{1}{\rho \sin \varphi} \frac{\partial u_{\varphi}}{\partial \theta} - \frac{u_{\theta}}{\rho} \right) \]

\[ \left( \frac{1}{\rho} \frac{\partial u_{\theta}}{\partial \varphi} \sin \varphi \frac{u_{\theta}}{\rho} + \frac{1}{\rho} \frac{\partial u_{\varphi}}{\partial \varphi} \sin \varphi \frac{u_{\theta}}{\rho} \right) \left( \frac{1}{\rho} \frac{\partial u_{\theta}}{\partial \varphi} - \cot \varphi \frac{u_{\theta}}{\rho} \frac{1}{\rho} \frac{\partial u_{\varphi}}{\partial \varphi} \sin \varphi \frac{u_{\theta}}{\rho} \right) \]

\[ + \frac{1}{\rho} \frac{\partial u_{\theta}}{\partial \varphi} \left( \frac{1}{\rho} \frac{\partial u_{\theta}}{\partial \varphi} + \sin \varphi \frac{u_{\theta}}{\rho} \sin \varphi + \cos \varphi \frac{u_{\theta}}{\rho} \right) \right] ; \]

\[ \tilde{\varepsilon}_{\theta \rho} = \frac{1}{2} \left( \frac{\partial u_{\theta}}{\partial \rho} + \frac{1}{\rho \sin \varphi} \frac{\partial u_{\varphi}}{\partial \theta} - \frac{u_{\theta}}{\rho} \right) + \frac{1}{2} \sin \varphi \left[ \frac{\partial u_{\theta}}{\partial \rho} \left( \frac{1}{\rho} \frac{\partial u_{\rho}}{\partial \theta} - \frac{u_{\theta}}{\rho} \sin \varphi \right) \right] + \left( \frac{\partial u_{\varphi}}{\partial \rho} \right) \left( \frac{1}{\rho} \frac{\partial u_{\varphi}}{\partial \theta} - \cos \varphi \frac{u_{\theta}}{\rho} \right) \left( \frac{1}{\rho} \frac{\partial u_{\varphi}}{\partial \theta} + \frac{u_{\rho}}{\rho} \right) \sin \varphi \cos \varphi \left( \frac{u_{\rho}}{\rho} \right) \]

\[ \times \left( \frac{1}{\rho} \frac{\partial u_{\theta}}{\partial \varphi} + \frac{u_{\theta}}{\rho} \sin \varphi + \cos \varphi \frac{u_{\theta}}{\rho} \right) \right] . \]

(11)

Expressions (6) and (11) simplify considerably if we neglect the non-linear terms in the components of displacement vectors and their derivatives (terms in brackets in (6) and (11)). In this case we obtain the components of the linear strain tensor in cylindrical and spherical coordinate systems. As before we denote those components by \( E_{rr}, ..., E_{zz}; E_{\rho \rho}, ..., E_{\theta \rho} \).
2.7 Compatibility conditions for linear and nonlinear strain tensor

The components of the small strain tensor are determined from (2.2-49), as

$$E_{kj} = \frac{1}{2} \left( \frac{\partial u_k}{\partial X_j} + \frac{\partial u_j}{\partial X_k} \right),$$

and because of symmetry, there are, in general, six independent $E_{ij}$. If the displacement vector $u$ is prescribed ($u_i$ are known continuously differentiable functions of $X_j$) then we can determine unique $E$ from (1). If, however, $E$ is known then we have six partial differential equations (1) that determine three components of the displacement vector $u_i$. It is obvious that the components of $E$ are not independent and that they must satisfy some additional conditions called the compatibility conditions that guarantee existence of the uniqueness of $u_i$. To determine the compatibility condition consider, in detail, the problem of determining the displacement vector $u$ from the strain tensor $E$. Since we are dealing with the continuum elasticity we assume that the body during deformation does not develop cracks or holes; that is, it remains continuous. Then, the components of the displacement vector $u$, that is, the functions $u_i = u(X_j)$, must be single valued functions of $X_j$. Thus we may reformulate our question as: what restrictions should be imposed on $E_{ij}(X_k)$ in order to ensure the existence of a unique continuous solution $u_i$ of (1)? Suppose we are given a point $P$ as a simply connected body. Suppose further that the displacement vector of the point $P$ is $u_P$ and is given as $u_P = u_P^i e_i$. We want to determine the displacement vector of an arbitrary point $Q$ (see Fig. 6).

---

6 A body is called simply connected if in the region $\kappa$ occupied by the body every closed curve contained within $\kappa$ (within the body) could be shrunk to a point by a continuous deformation, and that during this process the curve does not leave $\kappa$. 
By integrating (2.4-7) we obtain

$$u_i^Q = u_i^P + \int_P^Q [E_{ij} + \omega_{ij}] dX_j. \tag{2}$$

Partial integration of the second term under the integral sign in (2) leads to

$$\int_P^Q \omega_{ij} dX_j = \int_P^Q \omega_{ij} d(X_j - X_j^Q) = [\omega_{ij}(X_j - X_j^Q)]_P^Q + \int_P^Q \frac{\partial \omega_{ij}}{\partial X_m} dX_m, \tag{3}$$

or

$$\int_P^Q \omega_{ij} dX_j = \omega_{ij}^P (X_j^Q - X_j^P) + \int_P^Q (X_j^Q - X_j) \frac{\partial \omega_{ij}}{\partial X_m} dX_m, \tag{4}$$

so that (2) becomes

$$u_i^Q = u_i^P + \omega_{ij}^P (X_j^Q - X_j^P) + \int_P^Q (X_j^Q - X_j) \frac{\partial \omega_{ij}}{\partial X_m} dX_m. \tag{5}$$

Next we express $\frac{\partial \omega_{ij}}{\partial X_m}$ in terms of the derivatives of $E_{ij}$. Thus,

$$\frac{\partial \omega_{ij}}{\partial X_m} = \frac{\partial}{\partial X_m} \left( \frac{1}{2} \left( \frac{\partial u_i}{\partial X_j} - \frac{\partial u_j}{\partial X_i} \right) \right) = \frac{1}{2} \left( \frac{\partial^2 u_i}{\partial X_j \partial X_m} - \frac{\partial^2 u_j}{\partial X_i \partial X_m} \right)$$

$$= \frac{\partial}{\partial X_j} \left( E_{im} - \frac{1}{2} \frac{\partial u_m}{\partial X_i} \right) - \frac{\partial}{\partial X_i} \left( E_{jm} - \frac{1}{2} \frac{\partial u_m}{\partial X_j} \right)$$

$$= \frac{\partial E_{im}}{\partial X_j} - \frac{\partial E_{jm}}{\partial X_i}. \tag{6}$$

By using (6) in (5) we obtain

$$u_i^Q = u_i^P + \omega_{ij}^P (X_j^Q - X_j^P)$$

$$+ \int_P^Q \left[ E_{im} + (X_j^Q - X_j) \left( \frac{\partial E_{im}}{\partial X_j} - \frac{\partial E_{jm}}{\partial X_i} \right) \right] dX_m. \tag{7}$$

The value $u_i^Q$ must be independent of the integration path (it must be equal for paths (1), (2), and (3) shown in Fig. 6). The first two terms in (7) depend on the coordinates of points P and Q only and therefore are independent of the integration path. The integral in (7) is path independent if (Stokes theorem) the function under the integral sign satisfies

$$\frac{\partial \Lambda_{im}}{\partial X_i} = \frac{\partial \Lambda_{il}}{\partial X_m}, \tag{8}$$
2.7 Compatibility conditions for linear and nonlinear strain tensor

where

\[ \Lambda_{im} = E_{im} - X_j \left( \frac{\partial E_{im}}{\partial X_j} - \frac{\partial E_{jm}}{\partial X_i} \right). \] \hspace{1cm} (9)

By using the fact that

\[ \frac{\partial X_j}{\partial X_i} = \delta_{jl}, \] \hspace{1cm} (10)

we obtain from (9)

\[ \Lambda_{im,l} = E_{im,l} - \delta_{jl}(E_{im,j} - E_{jm,i}) - X_j(E_{im,jl} - E_{jm,il}); \]

\[ \Lambda_{il,m} = E_{il,m} - \delta_{jm}(E_{il,j} - E_{jl,i}) - X_j(E_{il,jm} - E_{jl,im}), \] \hspace{1cm} (11)

where, as before, \((\cdot)_i = \frac{\partial (\cdot)}{\partial X_i}\). Substituting (11) into (8), it follows that

\[ E_{im,l} - (E_{im,l} - E_{lm,i}) - X_j(E_{im,jl} - E_{jm,il}) \]

\[ = E_{il,m} - (E_{il,m} - E_{ml,i}) - X_j(E_{il,jm} - E_{jl,im}). \] \hspace{1cm} (12)

Since all \(X_m\) are independent we obtain from (12)

\[ E_{im,jl} + E_{jl,im} - E_{il,jm} - E_{jm,il} = 0. \] \hspace{1cm} (13)

The system (13) is known as the compatibility equations. It represents necessary and sufficient conditions for the existence of the unique displacement field. There are 81 equations (13). However, the number of independent equations that follow from (13) is much smaller. Note that (13) changes sign when the indices \(i\) and \(j\) and \(m\) and \(l\) are interchanged. It follows then that (13) is an identity for \(i = j\) and \(m = l\). Also (13) does not change when we interchange the following indices: 1) \(i\) and \(m\) and \(j\) and \(l\); 2) \(i\) and \(l\) and \(j\) and \(m\); 3) \(i\) and \(j\) and \(m\) and \(l\). Therefore there are only six equations in (13) that are neither repeated nor identically equal to zero. Those equations are obtained for \((imjl) = (1212); (2323); (3131); (1213); (2321); (3132)\). Thus (13) leads to

\begin{align*}
E_{11,22} + E_{22,11} - 2E_{12,12} & = 0; \\
E_{22,33} + E_{33,22} - 2E_{23,23} & = 0; \\
E_{33,11} + E_{11,33} - 2E_{31,31} & = 0; \\
E_{11,23} + E_{23,11} - E_{31,12} - E_{12,13} & = 0; \\
E_{22,31} + E_{31,22} - E_{12,23} - E_{23,21} & = 0; \\
E_{33,12} + E_{12,33} - E_{23,31} - E_{31,32} & = 0. \hspace{1cm} (14)
\end{align*}

Equation (5) is called the Cesaro integral for the displacement. In it the term outside the integral determines the infinitesimal "rigid body displacement." In other words when \(E_{ij}\) is given, the displacement vector is determined up to the infinitesimal rigid body displacement.
One may ask the following question: what are the compatibility conditions for the (finite) Lagrange-Green deformation tensor $\tilde{E}$? In other words, what are the conditions that guarantee the existence of a unique solution $u_i$ of the system of nonlinear partial differential equations

$$\frac{1}{2} \left( \frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} \right) + \frac{1}{2} \left( \frac{\partial u_m}{\partial X_i} \right) \left( \frac{\partial u_m}{\partial X_j} \right) = \tilde{E}_{ij}, \quad (15)$$

when the right-hand side contains known functions of coordinates. To answer this question it is convenient to present an independent argument that guarantees the integrability of (15). We start by observing that the existence of a unique displacement vector field $u$ is equivalent to the existence of functions $x_i(X_j), \; i = 1, 2, 3$ in (2.1-2) and their inverses (2.1-3). The function $x_i(X_j)$ generates a metric tensor in the deformed configuration that we denoted by $g_{ij}$ (see (2.2-7) and (2.2-11)). In the undeformed configuration $\kappa_0$ the metric tensor is $\delta_{ij}$ (the Kronecker delta) and corresponds to the Euclidean space. In the Euclidean space the curvature tensor is equal to zero. Therefore, if there exist functions $x_i(X_j) = X_i + u_i$ with properties as stated in Section 2.2, the curvature tensor in the deformed configuration $\kappa$ must also be equal to zero since in $\kappa$ the body occupies (again) a part of Euclidean space. The well-known Riemann theorem states that a symmetric tensor $g_{ij}$ is a metric tensor for Euclidean space if and only if it is nonsingular $(\det |g_{ij}| \neq 0)$ positive definite $(g_{ij}a_ia_j > 0$ for all $a_i \neq 0, \; a_j \neq 0)$ and the Riemann-Christoffel tensor formed from it vanishes identically. In our case (see (2.2-14)) the metric tensor is

$$g_{ij} = \left( \delta_{mi} + \frac{\partial u_m}{\partial X_i} \right) \left( \delta_{mj} + \frac{\partial u_m}{\partial X_j} \right). \quad (16)$$

The condition that the Riemann-Christoffel tensor $R_{krij}$ vanish, becomes

$$R_{krij} = \frac{1}{2} \left( \frac{\partial^2 g_{kj}}{\partial X_i \partial X_r} + \frac{\partial^2 g_{ir}}{\partial X_j \partial X_k} - \frac{\partial^2 g_{ij}}{\partial X_k \partial X_r} - \frac{\partial^2 g_{kr}}{\partial X_j \partial X_i} \right) = g^{qs}(\Lambda_{q,ir} \Lambda_{s,jk} - \Lambda_{q,ij} \Lambda_{s,kr}) = 0. \quad (17)$$

In (16) we used $\Lambda_{p,ms}$ to denote the Christoffel symbols of the first kind, based on the metric $g_{ij}$; that is,

$$\Lambda_{p,ms} = \frac{1}{2} \left( \frac{\partial g_{mp}}{\partial X_s} + \frac{\partial g_{ps}}{\partial X_m} - \frac{\partial g_{ms}}{\partial X_p} \right), \quad (18)$$

$^7$In tensor analysis it is shown that vanishing of the Reimann-Cristoffel tensor guarantees commutativity of the second covariant derivatives. For example, if $v_k$ is a covariant vector, then its covariant derivative is $v_{k;p} = \partial v_k/\partial X_p + v_m \Lambda_{mp}^k$, where the Christoffel symbols of the second kind are given by definition as $\Lambda_{ik}^j = g^{jm} \Lambda_{m,ik}$. The condition $R_{rkij} = 0$ implies that $(v_{k;r})_{,l} = (v_{k,l})_{,r}$. 
2.8 Plane state of strain

and $g^{qs}$ to denote the metric tensor of the dual basis.\(^8\) Note that comma in $\Lambda_{p,ms}$ does not denote differentiation but serves to separate indices ($m$ and $s$) with respect to which $\Lambda_{p,ms}$ is symmetric ($\Lambda_{p,ms} = \Lambda_{p,sm}$) from the index $p$. By substituting (16) into (17) and (18) we can obtain the compatibility conditions in terms of the derivatives of the displacement vector. It can be shown that (17) reduces to (13) if the terms of the order higher than the first $\partial u_i/\partial x_j$ are neglected.

We note that in the theory of the dislocations the integral of displacement vector over a closed curve (see Fig. 6) is not equal to zero; that is,

$$\left( \int_{P}^{Q} du_i \right) + \left( \int_{Q}^{P} du_i \right) = b_i. \quad (19)$$

In (19) we use $\left( \int_{P}^{Q} du_i \right)_t$ to denote the integral from the point P to the point Q along the path $t = 1, 3$. The vector $b = b_4 e_i$ is called the Burgers vector (see van Bueren (1960) for example). The condition (19) leads to the form of compatibility equations in which the right-hand side of (13) is not equal to zero. We are not concerned with such materials.

2.8 Plane state of strain

As a special case of the general deformation we present the result for the plane state of strain. It is defined as: a body is in a plane state of strain if the displacement vector of any point is parallel to a fixed plane, called the plane of deformation, and is independent of the distance of the point from the plane of deformation.

Suppose that the deformation vector at each point of the body is parallel to the $\vec{x}_1 - \vec{x}_2$ coordinate plane. Then from the definition it follows that

$$u_1 = u_1(X_1, X_2); \quad u_2 = u_2(X_1, X_2); \quad u_3 = 0. \quad (1)$$

From (1) we conclude that the strain tensor for the plane state of strain has the form

$$E = \begin{bmatrix} E_{11} & E_{12} & 0 \\ E_{12} & E_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (2)$$

where the components $E_{ij}$ are given as

$$E_{11} = \frac{\partial u_1}{\partial X_1}; \quad E_{22} = \frac{\partial u_2}{\partial X_2}; \quad E_{12} = \frac{1}{2} \left( \frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1} \right). \quad (3)$$

\(^8\)The metric tensor of the dual basis is determined from the condition

$$g_{ki}g^{km} = \delta^m_k \begin{cases} 1 & \text{if } m = k \\ 0 & \text{if } m \neq k \end{cases}.$$
An example of a long body (rod) that is loaded along its length is shown in Fig. 7.

If the load is independent of the axial coordinate $X_3$, then the body is in the plane state of strain. For a line element defined by unit vector $\mathbf{n}$ we have the relative elongation (see (2.3-5)) given by $e_n = E_{ij} n_i n_j$. Since the unit vector $\mathbf{n}$ could be expressed in terms of a single angle, (see Fig. 8)

$$\mathbf{n} = \cos \varphi \mathbf{e}_1 + \sin \varphi \mathbf{e}_2,$$  \hspace{1cm} (4)

it follows that the relative elongation is

$$e_n = E_{11} \cos^2 \varphi + E_{22} \sin^2 \varphi + E_{12} \sin 2\varphi.$$  \hspace{1cm} (5)

Similarly, the shear angle for the directions $\mathbf{n}$ and $\mathbf{m}$, where

$$\mathbf{m} = -\sin \varphi \mathbf{e}_1 + \cos \varphi \mathbf{e}_2,$$  \hspace{1cm} (6)

we obtain from (2.3-17) as

$$\frac{1}{2} \gamma_{nm} = -\frac{1}{2} (E_{11} - E_{22}) \sin 2\varphi + E_{12} \cos 2\varphi.$$  \hspace{1cm} (7)

Equation (5) could be written as

$$e_n = \frac{1}{2} (E_{11} + E_{22}) + \frac{1}{2} (E_{11} - E_{22}) \cos 2\varphi + E_{12} \sin 2\varphi.$$  \hspace{1cm} (8)
Expressions (7) and (8) are of the same form as (1.14-4) for the plane state of stress. It is therefore possible to obtain Mohr’s circle for plane strain on the basis of (7) and (8).

In the principal axes the strain tensor for the plane state of strain reads

\[
E = \begin{bmatrix}
e_1 & 0 & 0 \\
0 & e_2 & 0 \\
0 & 0 & 0
\end{bmatrix},
\]

where \(e_1, e_2, \text{ and } e_3 = 0\) are principal strains. If (9) is used, the expression (7), (8) becomes

\[
e_n = \frac{1}{2}(e_1 + e_2) + \frac{1}{2}(e_1 - e_2) \cos 2\varphi; \quad \frac{1}{2}\gamma_{nm} = -\frac{1}{2}(e_1 - e_2) \sin 2\varphi.
\]

Finally we state compatibility conditions for the plane state of strain. From (2) and (2.7-13) we obtain single relation

\[
\frac{\partial^2 E_{11}}{\partial X_2^2} + \frac{\partial^2 E_{22}}{\partial X_1^2} - 2\frac{\partial^2 E_{12}}{\partial X_1 \partial X_2} = 0.
\]

2.9 Linear strain tensor. Cubical dilatation

As stated before, the linear strain tensor is obtained if we neglect the quadratic terms in the displacement vector gradient. Then

\[
E_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} \right).
\]

Since

\[
x_i = X_i + u_i,
\]

we have

\[
\frac{\partial}{\partial X_i}(\cdot) = \frac{\partial}{\partial x_j}(\cdot) \frac{\partial x_j}{\partial X_i} = \left[ \delta_{ij} + \frac{\partial u_j}{\partial X_i} \right] \frac{\partial}{\partial x_j}(\cdot).
\]

When the terms of the second order in \(\partial u_i/\partial X_j\) are neglected and if \(u_i\) are small (compared to the dimensions of the body) we conclude from (3) that

\[
\frac{\partial}{\partial X_i}(\cdot) = \frac{\partial}{\partial x_i}(\cdot); \quad X_i = x_i.
\]

Equation (4) holds if the quantity (\(\cdot\)) is of the order \(\partial u_i/\partial X_j\). In this case there is no difference in Lagrange and Euler representation so that the Lagrange-Green and Euler-Almanasi strain tensor for the case when second-order terms in \(\partial u_i/\partial X_j\) are neglected, coincide

\[
\tilde{E}_{ij} = \bar{E}_{ij} = E_{ij}.
\]
The first invariant of the strain tensor (1) has an important property. Namely, it is connected with the cubical dilatation. To show this, consider an element of the body in the undeformed state, as shown in Fig. 9.

![Figure 9](image)

The volume of the parallelepiped in Fig. 9 is

$$dV = dX_1 dX_2 dX_3.$$  \hfill (6)

We now determine the volume of the same element in the deformed configuration $\kappa$. In it the parallelepiped remains parallelepiped but its edges are not orthogonal. In $\kappa$ the edges are $(\partial \mathbf{r}/\partial X_1)dX_1$, $(\partial \mathbf{r}/\partial X_2)dX_2$, $(\partial \mathbf{r}/\partial X_3)dX_3$, so that volume $dv$ is equal to the triple product

$$dv = \left( \frac{\partial \mathbf{r}}{\partial X_1} dX_1 \right) \cdot \left( \frac{\partial \mathbf{r}}{\partial X_2} dX_2 \right) \times \left( \frac{\partial \mathbf{r}}{\partial X_3} dX_3 \right).$$  \hfill (7)

Using the identity

$$[\mathbf{u} \times \mathbf{v} \cdot \mathbf{w}] [\mathbf{a} \times \mathbf{b} \cdot \mathbf{c}] = \det \begin{bmatrix} \mathbf{u} \cdot \mathbf{a} & \mathbf{u} \cdot \mathbf{b} & \mathbf{u} \cdot \mathbf{c} \\ \mathbf{v} \cdot \mathbf{a} & \mathbf{v} \cdot \mathbf{b} & \mathbf{v} \cdot \mathbf{c} \\ \mathbf{w} \cdot \mathbf{a} & \mathbf{w} \cdot \mathbf{b} & \mathbf{w} \cdot \mathbf{c} \end{bmatrix},$$  \hfill (8)

applied to (7), we obtain

$$(dv)^2 =$$

$$\det \begin{bmatrix} \frac{\partial \mathbf{r}}{\partial X_1} & \frac{\partial \mathbf{r}}{\partial X_1} & \frac{\partial \mathbf{r}}{\partial X_1} \\ \frac{\partial \mathbf{r}}{\partial X_2} & \frac{\partial \mathbf{r}}{\partial X_2} & \frac{\partial \mathbf{r}}{\partial X_2} \\ \frac{\partial \mathbf{r}}{\partial X_3} & \frac{\partial \mathbf{r}}{\partial X_3} & \frac{\partial \mathbf{r}}{\partial X_3} \end{bmatrix} (dX_1 dX_2 dX_3)^2.$$  \hfill (9)

By comparing (9) with (2.2-7) we conclude that elements in the determinant are components of the metric tensor $g_{ij}$. Therefore by using (2.2-11) we have

$$dv = \sqrt{g} dX_1 dX_2 dX_3 = \sqrt{g} dV,$$  \hfill (10)
2.9 Linear strain tensor. Cubical dilatation

where

\[
g = \det \begin{vmatrix}
1 + 2\tilde{E}_{11} & 2\tilde{E}_{12} & 2\tilde{E}_{13} \\
2\tilde{E}_{12} & 1 + 2\tilde{E}_{22} & 2\tilde{E}_{23} \\
2\tilde{E}_{13} & 2\tilde{E}_{23} & 1 + 2\tilde{E}_{33}
\end{vmatrix}.
\] (11)

By expanding (11) we obtain

\[
\frac{dv}{dV} = \sqrt{1 + 2\bar{I}_1 + 4\bar{I}_2 + 8\bar{I}_3},
\] (12)

where \(\bar{I}_i, i = 1, 2, 3\) are the strain invariants of the Lagrange-Green strain tensor. We define next the \textit{cubical dilatation} \(e_v\) as

\[
e_v = \frac{dv - dV}{dV},
\] (13)

so that

\[
e_v = \sqrt{1 + 2\bar{I}_1 + 4\bar{I}_2 + 8\bar{I}_3} - 1.
\] (14)

For the case when the gradient of the displacement vector is small, that is, \(|\partial u_i/\partial x_j| \approx 0\), it follows from (14) that

\[
e_v = E_{11} + E_{22} + E_{33};
\] (15)

that is, the \textit{cubical dilatation is equal to the first invariant of the linear strain tensor} \(E_{ij}\).

For the analysis of deformation of elastic bodies in the case when one dimension is much larger than the other two (rods) it may happen that the large displacement vectors lead to small deformation. In those cases \(E_{ij}\) is much smaller than \(\omega_{ij}\) (see decomposition (2.2-51)). Then by using (2.2-51) we have

\[
\bar{E}_{ij} = E_{ij} + \frac{1}{2}\omega_{ki}\omega_{kj}.
\] (16)

From (16) with (2.4-13) we obtain

\[
\begin{align*}
\bar{E}_{11} & \approx E_{11} + \frac{1}{2}(w_2^2 + w_3^2); & \bar{E}_{12} & \approx E_{12} - \frac{1}{2}w_1w_2; \\
\bar{E}_{22} & \approx E_{22} + \frac{1}{2}(w_3^2 + w_1^2); & \bar{E}_{23} & \approx E_{23} - \frac{1}{2}w_2w_3; \\
\bar{E}_{33} & \approx E_{33} + \frac{1}{2}(w_1^2 + w_2^2); & \bar{E}_{13} & \approx E_{13} - \frac{1}{2}w_1w_3.
\end{align*}
\] (17)

The strain tensor given by (17) is used in rod and plate theories.
2.10 Measurement of strain. Strain gauges

Often it is important to measure the strains and stresses at the given point of an elastic body. The variety of techniques that are used for this purpose constitutes the subject of experimental stress analysis. The main techniques in use are: brittle lacquers, strain gauges, photoelasticity, photoelastic coatings, wave propagation techniques (acoustoelasticity), grid procedure (Moiré technique), X-ray techniques, etc. We present elements of strain measurements at the point by strain gauges.

The strain gauge (the electrical resistance strain gauge) is simply a length of wire formed into the shape shown in Fig. 10 and cemented on a nonconductive backing. The gauge is then bonded to the surface where we want to measure strain. The elongation of the wire causes change in electrical resistance and this is used to estimate the strain.

![Figure 10](image)

To determine the principal strains and the orientation of principal axes (see (2.8-9)) the strain rosette (in Fig. 11 we show an equiangular rosette) could be used. It consists of three strain gauges measuring strains in three directions at the point. From the readings of strain gauges rosettes we obtain the strains in three directions at a given point of an elastic body. The dimensions of the strain rosette are small so that we assume that all three gauges are positioned at a single point of the body. In general the choice of a suitable gauge requires consideration of many factors (physical size and form, resistance and sensitivity, operating temperature, strain limits, etc.).

![Figure 11](image)
Suppose that a strain rosette is arbitrarily oriented with respect to the principal directions of the strain tensor at a given point. Let $\Psi$ denote the unknown angle between the specific gauge, say “0”, and the $\bar{x}_1$ axis of the principal coordinate system $\bar{x}_1 - \bar{x}_2$. Let $\alpha$ be the angle between the “0” gauge and two other gauges, that we denote by “$+\alpha$” and “$-\alpha$” (see Fig. 12). We denote by $e_{-\alpha}$, $e_0$, and $e_\alpha$ the readings of the gauges “$-\alpha$,” “0,” and “$+\alpha$.” Then, from (2.8-10)$_1$ we have

$$e_{-\alpha} = \frac{1}{2}(e_1 + e_2) + \frac{1}{2}(e_1 - e_2) \cos(2\Psi - 2\alpha);$$

$$e_0 = \frac{1}{2}(e_1 + e_2) + \frac{1}{2}(e_1 - e_2) \cos 2\Psi;$$

$$e_\alpha = \frac{1}{2}(e_1 + e_2) + \frac{1}{2}(e_1 - e_2) \cos(2\Psi + 2\alpha), \quad (1)$$

where $e_1$ and $e_2$ are (unknown) principal strains.

![Figure 12](image)

The system (1) has to be solved for $e_1$, $e_2$, and $\Psi$. To do this we introduce the following notation

$$A = \frac{1}{2}(e_1 + e_2); \quad B = \frac{1}{2}(e_1 - e_2) \cos 2\Psi; \quad C = \frac{1}{2}(e_1 - e_2) \sin 2\Psi. \quad (2)$$

With (2), we have

$$e_{-\alpha} = A + B \cos 2\alpha + C \sin 2\alpha; \quad e_0 = A + B; \quad e_\alpha = A + B \cos 2\alpha - C \sin 2\alpha. \quad (3)$$

From (3) it follows that

$$e_{-\alpha} - e_\alpha = 2C \sin 2\alpha;$$

$$2e_0 - e_{-\alpha} - e_\alpha = 2B(1 - \cos 2\alpha), \quad (4)$$
or
\[ \frac{e_{-\alpha} - e_\alpha}{2e_0 - e_{-\alpha} - e_\alpha} = \frac{C}{B} \frac{\sin 2\alpha}{1 - \cos 2\alpha}. \] (5)

However, from (2) we also have
\[ \frac{C}{B} = \tan 2\Psi, \] (6)
so that by combining (5) and (6) we obtain
\[ \tan 2\Psi = \frac{1 - \cos 2\alpha}{\sin 2\alpha} \frac{e_{-\alpha} - e_\alpha}{2e_0 - e_{-\alpha} - e_\alpha}. \] (7)

The expression (7) determines the orientation of the principal axes. The principal values of the strain tensor we determine as follows. By adding and subtracting the expressions (2) and (3) we have
\[ \frac{1}{2}(e_1 + e_2) = A = e_0 - B = e_0 - \frac{2e_0 - e_{-\alpha} - e_\alpha}{2(1 - \cos 2\alpha)}, \]
\[ = \frac{e_{-\alpha} + e_\alpha - 2e_0 \cos 2\alpha}{2(1 - \cos 2\alpha)}, \] (8)

and
\[ \frac{1}{2}(e_1 - e_2) = \frac{C}{\sin 2\Psi} = \frac{e_{-\alpha} - e_\alpha}{2\sin 2\alpha \sin 2\Psi}. \] (9)

Finally, by combining (8) and (9), we get
\[ e_{1,2} = \frac{e_{-\alpha} + e_\alpha - 2e_0 \cos 2\alpha}{2(1 - \cos 2\alpha)} \pm \frac{e_{-\alpha} - e_\alpha}{2\sin 2\alpha \sin 2\Psi}. \] (10)

Since $\psi$ is determined from (7) the values of $e_1$ and $e_2$ follow from (10).

As can be seen, the shear angle could not be measured directly by the strain rosette. We present now a procedure that allows determination of the shear angle between two directions defined by unit vectors $\mathbf{n}$ and $\mathbf{m}$ (see Fig. 13). Suppose that $\theta = \angle(\mathbf{n}, \mathbf{m})$ is known. Suppose further that the unit vectors $\mathbf{a}$ and $\mathbf{b}$ are defined so that $\mathbf{a}$ bisects $\theta$ and that $\mathbf{b}$ is orthogonal to $\mathbf{a}$. Then
\[ 2\mathbf{a}\cos(\theta/2) = \mathbf{m} + \mathbf{n}; \quad 2\mathbf{b}\sin(\theta/2) = \mathbf{m} - \mathbf{n}; \quad \mathbf{a} \cdot \mathbf{b} = 0. \] (11)
From (11) it follows that
\[ \mathbf{m} = \mathbf{a} \cos(\theta/2) - \mathbf{b} \sin(\theta/2); \quad \mathbf{n} = \mathbf{a} \cos(\theta/2) + \mathbf{b} \sin(\theta/2). \] (12)
By using (12) in (2.3-16) we obtain
\[ \gamma_{nm} \sin \theta = E_{ij}(a_i a_j - b_i b_j)(1 - \cos^2 \theta), \] (13)
or
\[ \gamma_{nm} = (e_a - e_b) \sin \theta. \] (14)
From (14) it follows that by setting two strain gauges along the unit vectors \( \mathbf{a} \) and \( \mathbf{b} \) and by measuring \( e_a \) and \( e_b \) we can determine \( \gamma_{nm} \). The expression (13) was derived by Boulanger and Hayes (1992).

We make a comment concerning the accuracy of the strain gauges measurements. Strictly speaking the strain \( e_{\text{body}} \) at the point of the body where the strain gauge is positioned is not equal to the strain \( e_{\text{gauge}} \) in the strain gauge unless the extensional stiffness of the gauge is very small compared to the extensional stiffness of the body. This is very important to know when strain gauges are used to measure strain in materials with small extensional stiffness (such as plastics). By using rather complicated analysis of the coupled deformation (deformation of the body where the strain is measured and deformation of strain gauge) in Alexandrov and Mihatrayn (1983) it was shown that
\[ k = \frac{e_{\text{body}}}{e_{\text{gauge}}} = \frac{4}{\pi d} \left(1 + \frac{d}{2}\right), \] (15)
where \( d \) is a constant determined by the material and the stress state in the body and dimensions and material of the gauge. Consider a strain gauge of length \( 2a \) and width \( b \) positioned on a body whose modulus of elasticity and Poisson’s ratio (see Section 3.4) are \( E_2 \) and \( \nu_2 \). Then \( d \) is given as
\[ d = \frac{a E_2}{2(1 - \nu_2^2) E_g}; \quad \text{or} \quad d = \frac{a E_2}{2} \frac{1}{E_g}. \] (16)
The first expression in (16) corresponds to the plane state of strain and the second to the plane state of stress. Also in (16) we used \( E_g \) to denote the extensional stiffness of the gauge per unit of its width. Suppose that the gauge of length \( 2a \) and width \( b \) is loaded by an extensional force \( Q \). Suppose further that this force produces elongation of the gauge equal to \( \Delta l \). Then the gauge stiffness is \( E_g = Q(2a)/|b(\Delta l)| \). The constant \( k \) may, in certain measurements, have value as high as 30. If \( d > 10 \), it may be assumed that \( k = 1 \).

Problems

1. For the displacement vector with components
\[ u_1 = kX_1; \quad u_2 = k(X_2 + 4X_3); \quad u_3 = k(4\sqrt{2}X_1 + 3X_3), \]
determine the strain tensor, principal extensions, principal directions, and cubical dilatation.

2. Consider the strain tensors given by

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & k(X_2^2 + X_3^2) & kX_2X_3 \\
0 & kX_2X_3 & kX_3^2
\end{bmatrix},
\]

i) \[ [E_{ij}] = \]

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & kX_1(x_2^2 + X_3^2) & kX_1X_2X_3 \\
0 & kX_1X_2X_3 & kX_1^2X_3^2
\end{bmatrix},
\]

where \( k = \text{const.} \) Is there a unique displacement field that could be determined from \( E_{ij} \)?

3. The displacement field corresponding to the bending problem of plates is given as

\[
U_1 = u_0(X_1, X_2) + X_3\psi(X_1, X_2), \quad U_2 = v_0(X_1, X_2) + X_3\varphi(X_1, X_2); \quad U_3 = w_0(X_1, X_2),
\]

where \( u_0, v_0, w_0, \psi, \) and \( \varphi \) are continuously differentiable functions. Determine the components \( E_{ij} \).

4. Derive expressions (2.10-7) and (2.10-10) for the case when the angles between the “0” and two other gauges are not equal.

5. The readings on the strain rosette are

\[
e_0 = 308 \cdot 10^{-6}; \quad e_{45} = -432 \cdot 10^{-6}; \quad e_{-45} = 592 \cdot 10^{-6}.
\]

Determine the principal extensions and the orientation of principal axes.

6. The components of a strain tensor are

\[
E_{11} = 2X_1^2 + 3X_2^2 + X_3 + 1; \quad E_{22} = 2X_2^2 + X_1^2 + 3X_3 + 2; \quad E_{33} = 3X_1 + 2X_2 + X_3^2 + 1; \quad E_{12} = 4X_1X_2; \quad E_{13} = E_{23} = 0.
\]

Are the compatibility conditions satisfied? Determine the rotation tensor \( \omega \).

7. The displacement vector has the components given by

\[
u_1 = kX_1^2, \quad u_2 = kX_2X_3, \quad u_3 = k(2X_1X_3 + X_2^2),
\]

where \( k = \text{const.} \) Determine the Lagrange-Green strain tensor \( \bar{E} \). At the point \( T(1, 1, -1) \) determine the relative extension \( e_n \) for the case when \( n = e_2 \). Calculate the components of the (linear) strain tensor \( E \) and the relative extension according to the expression (2.3-5) for the same direction \( n \).
8. Check the compatibility conditions for the following strain tensors

a) \( E_{11} = k(X_1^2 + X_2^2 + X_3^2), \ E_{12} = 2kX_1X_2X_3, \ E_{22} = X_1^2 + X_2^2, \ E_{33} = X_3^2, \ E_{13} = E_{23} = 0 \);

b) \( E_{11} = k(X_1^2 + X_2^2), \ E_{12} = 2kX_1X_2X_3, \ E_{22} = X_2^2X_3, \ E_{13} = E_{23} = E_{33} = 0, \)

where \( k = \text{const} \).

9. Starting from (2.2-21) and by using (1.6-6) show that

\[ X_mX_m = X_1^2 + X_2^2 + X_3^2 = X_m^*X_m^* = (X_1^*)^2 + (X_2^*)^2 + (X_3^*)^2; \]

that is, the length of the vector \( \mathbf{R} \) does not change under the transformation \( \mathbf{Q} \).

10. A plate \( 0 \leq X_1 \leq a, \ 0 \leq X_2 \leq b \) is fixed along the sides \( X_1 = 0 \) and \( X_2 = 0 \). The other two sides are free. If the components of the strain tensor are

\[ E_{11} = k(X_1^2X_2 + X_2^2), \quad E_{22} = kX_1X_2^2, \]

where \( k = \text{const} \), determine the components \( u_1 \) and \( u_2 \) of the displacement vector.

11. In the expression (2.4-18) the rigid body motion corresponds to the case when \( \mathbf{E} = 0 \) (\( \partial u_i/\partial x_j + \partial u_j/\partial x_i = 0 \)) so that the displacement vector of the point \( Q \) with respect to the point \( P \) is given as

\[ \mathbf{u}(Q) = \mathbf{u}(P) + \mathbf{w} \times d\mathbf{X}. \quad (a) \]

Suppose that the distance between \( P \) and \( Q \) is finite (instead of \( d\mathbf{X} \) we have \( \mathbf{a} \), with \( \mathbf{a} = a_i\mathbf{e}_i \) and \( a_i \) finite) so that

\[ \mathbf{u}(Q) = \mathbf{u}(P) + \mathbf{w} \times \mathbf{a}. \quad (b) \]

Show that the Lagrange-Green strain tensor \( \bar{\mathbf{E}} \) corresponding to \( (b) \) is not equal to zero and that it is given by the following expression

\[ [\bar{E}_{ij}] = \frac{1}{2} \begin{bmatrix} w_2^2 + w_3^2 & -w_1w_2 & -2w_1w_3 \\ -w_1w_2 & w_1^2 + w_2^2 - w_2w_3 & -w_1w_3 \\ -w_1w_3 & -w_2w_3 & w_1^2 + w_2^2 \end{bmatrix}. \]

Therefore \( (b) \) determines displacement field "as a rigid body" only if \( \mathbf{w} \) is small so that \( |w_iw_j| \approx 0 \).

12. The orthogonal matrix \( (\mathbf{Q}^T = \mathbf{Q}^{-1}) \) defined in (1.6-11) corresponds to a rotation about a fixed axis. Show that:
i) any orthogonal matrix has proper values of unit magnitude;

ii) since \( \det |c_{ij}| = \pm 1 \) it follows that the proper values of an orthogonal matrix are either all real of multiplicity 2 or 3, or one proper value is real and the others are complex conjugates;

iii) the principal direction corresponding to the real proper value is called the axis of rotation. Therefore the unit vector \( \mathbf{d} \) along the axis of rotation satisfies

\[
\mathbf{Qd} = \mathbf{d},
\]

iv) the angle of rotation \( \phi \) about the axis of rotation with unit vector \( \mathbf{d} \) with components \( d_i \) are determined from

\[
\cos \phi = \frac{1}{2} [(c_{11} + c_{22} + c_{33}) - 1];
\]

\[
d_i = \frac{1}{2 \sin \phi} \epsilon_{ijk} c_{jk},
\]

where \( \epsilon_{ijk} \) is the Levi-Civita permutation symbol

\[
\epsilon_{ijk} = \begin{cases} 
1 & \text{for } i, j, k = 1, 2, 3; 2, 3, 1; 3, 1, 2 \\
-1 & \text{for } i, j, k = 1, 3, 2; 3, 2, 1; 2, 1, 3 \\
0 & \text{for } i = j; \ i = k; \ j = k;
\end{cases}
\]

v) if the vector \( \mathbf{d} \) and the angle of rotation \( \phi \) are known, then the elements of the matrix \( \mathbf{Q} \) are

\[
c_{ij} = \cos \phi d_i d_j - (1 + \cos \phi)d_i d_j - \sin \phi \epsilon_{ijk} d_k.
\]

13. Starting from (2.6-4) and using (2.4-7) in the form

\[
du = [E_{rr} dr + (E_{r\theta} + \omega_{r\theta})(rd\theta) + (E_{rz} + \omega_{rz})dz]e_r,
\]

\[
+ [(E_{\theta r} + \omega_{\theta r}) dr + E_{\theta \theta}(rd\theta) + (E_{\theta z} + \omega_{\theta z})dz]e_\theta \quad (a)
\]

\[
+ [(E_{z r} + \omega_{z r}) dr + (E_{z \theta} + \omega_{z \theta})(rd\theta) + E_{zz} dz]e_z,
\]

determine the components of the infinitesimal rotation tensor in the cylindrical coordinate system. Also by starting from (2.6-8) and by using

\[
du = [E_{\rho\rho} d\rho + (E_{\rho\theta} + \omega_{\rho\theta})(\rho \sin \varphi d\theta) + (E_{\rho\varphi} + \omega_{\rho\varphi})(\rho d\varphi)]e_\rho
\]

\[
+ [(E_{\theta\rho} + \omega_{\theta\rho}) d\rho + E_{\theta\theta}(\rho \sin \varphi d\theta) + (E_{\theta\varphi} + \omega_{\theta\varphi})(\rho d\varphi)]e_\theta
\]

\[
+ [(E_{\varphi\rho} + \omega_{\varphi\rho}) d\rho + (E_{\varphi\theta} + \omega_{\varphi\theta})(\rho \sin \varphi d\theta) + E_{\varphi\varphi}(\rho d\varphi)]e_\varphi,
\]

determine the components of the rotation tensor in the spherical coordinate system (for answers see Section 4.3)
14. Suppose the coordinates \( x_i \) of an arbitrary point in the deformed state are given in terms of its coordinates in the undeformed state \( X_j \) by

\[
x_i = c_{ij} X_j + b_i,
\]

where \( c_{ij} \) are components of an orthogonal tensor \( (Q = [c_{ij}], \quad Q^T = Q^{-1}) \) and \( b_i \) are constants. By using \( u_i = x_i - X_i \) show that

\[
\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_m}{\partial X_i} \frac{\partial u_m}{\partial X_j} = 0.
\]

Thus, the rigid body motion corresponds to \( \tilde{E}_{ij} = 0 \) (see (2.2-16)).

15. By using the permutation symbol defined in Problem 12, show that:

i) The compatibility equations (2.7-13) can be written as

\[
\epsilon_{ikl} \epsilon_{jms} \frac{\partial^2 E_{ls}}{\partial X_k \partial X_m} = 0. \tag{a}
\]

ii) Next by using the identity \( \epsilon_{ijk} \epsilon_{klm} = \delta_{i}^{\alpha} \delta_{j}^{\beta} \delta_{km}^{\gamma} - \delta_{im}^{\alpha} \delta_{mj}^{\beta} \) show that (a) can be transformed into

\[
\frac{\partial^2 E_{ji}}{\partial X_k \partial X_k} + \frac{\partial^2 E_{kk}}{\partial X_i \partial X_j} - \left( \frac{\partial^2 E_{ik}}{\partial X_k \partial X_i} + \frac{\partial^2 E_{ki}}{\partial X_j \partial X_i} \right)
\]

\[
+ \left( \frac{\partial^2 E_{kl}}{\partial X_k \partial X_l} - \frac{\partial^2 E_{kk}}{\partial X_l \partial X_l} \right) \delta_{ij} = 0. \tag{b}
\]
Chapter 3
Hooke’s Law

3.1 Introduction

The analysis presented so far applies to all bodies that could be, with sufficient accuracy, described as continuous bodies. Since we intend to study elastic bodies, we now present an experimental foundation that serves as a basis for mathematical description of elastic bodies. The first experiments on elastic bodies were designed to measure the force needed to break a rod or rope made of a specific material. We mention Galileo’s (1638) apparatus for breaking a beam in bending with an end load, Da Vinci’s (1680) apparatus to measure the force needed to rupture a rope, and Mariotte’s (1700) apparatus for measuring the force needed to break an elastic beam by extension.1

In Fig. 1 an experimental apparatus suggested by Leonardo da Vinci is shown. Sand flows from a large container into a special container (P). At the moment when the rope breaks, the flow of sand is stopped. Then, the weight of sand in the container (P) is measured. It corresponds to the force needed for rupture of the rope. Mariotte’s apparatus is shown in Fig. 2a. It is designed to measure the force needed to break the rod AB by applying weight denoted by C.

The relation between the stress and deformation, in continuum mechanics, is called the constitutive equation. G.W. Leibniz was the first to realize that the constitutive equations must be determined experimentally.2 This was stated after a series of experiments performed by Robert Hooke in 1676. Hooke published (at the end of a paper on helioscopes) an anagram in the following form: ceiïnossttvu. Later in his book De Potentia Restitu­ativa, published in 1678, Hooke gave the solution of the anagram: Ut tensio sic vis.

As the experimental basis for his law Hooke refers to four specific types of experiments that he had performed in making his discovery: the deflection of a metal wire in the form of a helical spring, the twist of a spiral spring,

---

1 For a historical account of early experiments in solid mechanics with detailed references, see Bell (1973).

the tensile deformation of a long metal wire, and the end deflection of a cantilever wooden beam. In Fig. 2b the apparatus for three of Hooke's experiments are shown schematically.

Figure 1

To explain the experimental foundation of Hooke's law we analyze the relation connecting stress and deformation (strain) for the simple extension experiment. The type of loading and the characteristic results are shown in Fig. 3. Let $A_0$ denote the cross-sectional area of the rod in the undeformed state.

Figure 2

Suppose that the stress state in the rod is linear and that the stress vector is constant across the cross-section. Then we define $\sigma_R = F/A_0$. 
The value $\sigma_R$ is called engineering or Piola-Kirchhoff stress. It measures the actual force when it is taken per unit area of the reference (undeformed) configuration. We obtain the Cauchy stress $\sigma$ (as defined in Chapter 1) if the force $F$ is divided by the cross-sectional area in the actual (deformed) configuration. If we determine, experimentally, for each value of the force $F$ the length of the rod $l$, then we can calculate the strain (relative elongation, see 2.2-30) as $e = (l - l_0)/l_0$, where $l_0$ is the length of the rod in the undeformed configuration. From Fig. 3 it can be seen that, at least at the beginning of deformation, the dependence of $\sigma_R$ on $e$ can be expressed in the form of Hooke’s law

$$\sigma_R = E e,$$

where $E$ is a constant called the Young modulus. The relation (1) is valid up to the point P in Fig. 3 which is called the proportionality limit.

Equation (1) is a special case of a more general relation

$$\sigma_R = f(e),$$

where $f$ is a nonlinear function. Since the strain $e$ could be determined from the strain tensor $E$, it follows that, at least for the case shown in Fig. 3, that the stress $\sigma_R$ is a function of the strain tensor. With this as a motivation, we assume that there is a functional relation between the Cauchy stress tensor and the strain tensor; that is,

$$\sigma_{ij} = F_{ij}(E_{rs}),$$

where the functions $F_{ij}$ satisfy

$$F_{ij}(E_{rs} = 0, r, s = 1, 2, 3) = 0.$$

The condition (4) is the mathematical expression of the assumption that in the undeformed state the stress in the body is equal to zero. We assume further that $F_{ij}$ are analytic functions of all of its arguments so that we
can expand each in a Taylor series. Retaining the first-order terms only, we obtain

\[ \sigma_{ik} = C_{iklm}E_{lm}, \]  

where we use \( C_{iklm} \) to denote constants. Those constants form a four-index system called elasticities. Elasticities transform, by transformation of the coordinate system, in a way that corresponds to the fourth-order tensors. Therefore the elasticities form the \textit{elasticity tensor}. This tensor has 81 components but they are not all independent. The expression (5) represents the \textit{generalized Hooke's law}. If \( C_{iklm} \) are independent of the position, the material is called \textit{elastically homogeneous} material. In the next two sections we examine the quantities \( C_{iklm} \) in (5).

### 3.2 Transformation of the elasticity tensor by rotation of coordinate system

Suppose that in the Cartesian coordinate system \( \bar{x}_i, i = 1, 2, 3 \) the generalized Hooke's law is given in the form

\[ \sigma_{ij} = C_{ijkl}E_{kl}. \]  

If \( x_i^*, i = 1, 2, 3 \) are axes of another Cartesian coordinate system we want to find the form that (1) takes in \( x_i^*, i = 1, 2, 3 \) and thus to determine the transformation law for the quantities \( C_{ijkl} \). Let \( Q \) be the orthogonal matrix with components \( c_{ij} \) (see Section 1.6)

\[ c_{ij} = \cos \angle(x_i^*, \bar{x}_j). \]  

Suppose that in the system \( x_i^* \) we have

\[ \sigma_{kl}^* = C_{klst}E_{st}^*. \]  

The transformation law for \( \sigma_{ij} \) and \( E_{kl} \) are known (see (1.6-10)) and (2.2-24))

\[ \sigma_{kl}^* = \sigma_{mn}c_{km}c_{ln}^*; \quad E_{st}^* = E_{pr}c_{sp}c_{tr}. \]  

From (4)2 it follows that \( (E^* = QE^T \Rightarrow E = Q^T E^* Q) \),

\[ E_{pr} = E_{st}^*c_{sp}c_{tr}. \]  

By substituting (5) into (1) and the so-obtained result in (4)1 we have

\[ \sigma_{kl}^* = C_{mnpq}E_{st}^*c_{sp}c_{tr}c_{km}c_{ln}. \]  

By comparing (6) with (3) it follows that

\[ C_{klst} = C_{mnpq}c_{km}c_{ln}c_{sp}c_{tr}. \]
From equation (7) we conclude that the elasticities constitute a Cartesian tensor of rank four since any fourth-order system that satisfies (7) is, by definition, a Cartesian tensor of rank four. This tensor is called the elasticity tensor. Note also that (7) guarantees that the expression (1) is coordinate indifferent; that is, it has the same form in all Cartesian coordinate systems.

3.3 Anisotropic, orthotropic, and isotropic elastic body

We now study some special classes of (3.2-1) that correspond to specific elastic materials. For each of these materials we specify the form of the elasticity tensor $C_{ijkl}$.

Suppose a body is deformed so that $E_{12} = E_{21} \neq 0$ and all other components of the strain tensor are equal to zero. Then, from (3.2-1), we have

$$\sigma_{mn} = C_{mn12}E_{12} + C_{mn21}E_{21} = [C_{mn12} + C_{mn21}]E_{12},$$

(1)

where we used the symmetry property of the strain tensor. We define now the fourth-order tensors

$$\tilde{C}_{mnkl} = \frac{1}{2}(C_{mnkl} + C_{mnlk}); \quad \tilde{C}_{mnkl} = \frac{1}{2}(C_{mnkl} - C_{mnlk}).$$

(2)

The tensor $\tilde{C}_{mnkl}$ is symmetric and $\tilde{C}_{mnkl}$ is skew-symmetric with respect to the last two indices and $C_{nmkl} = \tilde{C}_{nmkl} + \tilde{C}_{nmkl}$. With (2) equation (1) can be written as

$$\sigma_{mn} = \tilde{C}_{mn12}E_{12} + \tilde{C}_{mn21}E_{21}. $$

(3)

By similar consideration (with other $k, l$) we can prove that by using (2) the elasticity tensor could be made symmetric with respect to the last two indices; that is, $\tilde{C}_{ijkl} = \tilde{C}_{ijlk}$. In what follows we assume that $C_{ijkl}$ is symmetrized and we omit the bar over $\tilde{C}_{ijkl}$ in (2). Then in the generalized Hooke's law given by (3.2-1) we have

$$C_{ijkl} = C_{ijlk}. $$

(4)

From (4) it follows that the tensor $C_{ijkl}$ has at most $3\cdot3\cdot6 = 54$ independent constants.

Suppose now that the deformation at the point is such that the only nonzero strain tensor component is $E_{11} \neq 0$. Then (3.2-1) leads to

$$\sigma_{mn} = C_{mn11}E_{11}. $$

(5)

From (5) it follows that

$$\sigma_{12} = C_{1211}E_{11}; \quad \sigma_{21} = C_{2111}E_{11}. $$

(6)
Since $\sigma_{12} = \sigma_{21}$, we have
\[ C_{1211} = C_{2111}, \tag{7} \]
or, in general,
\[ C_{ijkl} = C_{jikl}. \tag{8} \]
From (8) it follows that $C_{ijkl}$ has only 36 independent components. Further reduction of the number of independent components of $C_{ijkl}$ can be achieved by thermodynamic considerations. We begin this analysis by introducing the strain energy density. Consider the work $dU$ done by the stress components $\sigma_{ij}$ acting on a unit cube of elastic material when the deformation is increased so that the strain tensor components increase by $dE_{ij}$. This work is given as (see Section 7.2)
\[ dU = \sigma_{ij} dE_{ij}. \tag{9} \]
The quantity $U = \int_{E_{ij}=0}^{E_{ij}} \sigma_{ij} dE_{ij}$ defined by its differential in (9) is called the strain energy density. Note that from (9) we have the following relation
\[ \sigma_{ij} = \frac{\partial U}{\partial E_{ij}}. \tag{10} \]
The strain energy density $U$ plays an important role in elasticity theory.\(^3\) The existence of the function $U$ defined by (9) is guaranteed for reversible isothermal or adiabatic deformation process. If we introduce the expression (see Section 7.2 for motivation)
\[ dU_c = E_{ij} d\sigma_{ij}, \tag{11} \]
then,
\[ dU + dU_c = d(\sigma_{ij} E_{ij}), \tag{12} \]
and
\[ E_{ij} = \frac{\partial U_c}{\partial \sigma_{ij}}. \tag{13} \]
The quantity $U_c$ is called the complementary energy density and the relation (see (12))
\[ U_c = \sigma_{ij} E_{ij} - U; \tag{14} \]
is called Legendre transformation. By introducing (3.2-1) in (9) we obtain
\[ dU = C_{ijkl} E_{kl} dE_{ij} = \frac{\partial U}{\partial E_{ij}} dE_{ij}, \tag{15} \]
\(^3\)The strain energy density $U$ was introduced by G. Green in 1839 in his study of light refraction of noncrystallized media.
and
\[ \frac{\partial U}{\partial E_{ij}} = C_{ijkl} E_{kl}. \] (16)

By differentiating (16) it follows that
\[ \frac{\partial}{\partial E_{kl}} \left( \frac{\partial U}{\partial E_{ij}} \right) = C_{ijkl}. \] (17)

Interchanging the order of differentiation in (17), we obtain
\[ \frac{\partial}{\partial E_{ij}} \left( \frac{\partial U}{\partial E_{kl}} \right) = C_{klij}. \] (18)

From (17) and (18) it follows that
\[ C_{ijkl} = C_{klij}. \] (19)

We represent the tensor $C_{ijkl}$, with the properties (4), (8), and (19) in the matrix form
\[
[C_{ijkl}] = \begin{bmatrix}
C_{1111} & C_{1122} & C_{1133} & C_{1112} & C_{1113} & C_{1123} \\
C_{2211} & C_{2222} & C_{2233} & C_{2212} & C_{2213} & C_{2223} \\
C_{3311} & C_{3322} & C_{3333} & C_{3312} & C_{3313} & C_{3323} \\
C_{1211} & C_{1222} & C_{1233} & C_{1212} & C_{1213} & C_{1223} \\
C_{1311} & C_{1322} & C_{1333} & C_{1312} & C_{1313} & C_{1323} \\
C_{2311} & C_{2322} & C_{2333} & C_{2312} & C_{2313} & C_{2323}
\end{bmatrix}. \] (20)

The symmetry condition (19) implies that (20) has only 21 independent components. The material with 21 independent constants in $C_{ijkl}$ is called anisotropic material.

We now begin the study of special classes of materials by specifying the form of $C_{ijkl}$ for each class.

**Monoclinic material** is a material that exhibits symmetry with respect to one plane. We take, without loss of generality, this plane to be the $\bar{x}_1 - \bar{x}_2$ coordinate plane. Then elastic properties of material do not change under the coordinate change
\[ x_1^* = x_1; \quad x_2^* = x_2; \quad x_3^* = -x_3. \] (21)

The transformation matrix for (21) reads
\[ Q_1 = [c_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \] (22)

---

4There has been a long controversy about the number of independent components of $C_{ijkl}$. Poisson and Cauchy, on the basis of a (simplified) molecular model of an elastic body claimed that there are only 15 independent constants. G. Green obtained the correct number 21. Much later M. Born by using modern results of molecular theory proved that the number of independent constants according to molecular theory is also 21.
Under the transformation (22) we must have
\[ C_{ijkl}^* = C_{ijkl}. \] (23)

By using (22) in (3.2-7) we obtain, for example,
\[ C_{1111}^* = C_{1111}. \] (24)

Therefore the condition (23) is satisfied for arbitrary \( C_{1111}. \) However (3.2-7) leads also to
\[ C_{1123}^* = -C_{1123}, \] (25)

so that
\[ C_{1123} = 0. \] (26)

By similar consideration we can show that 7 additional constants are equal to zero so that for monoclinic materials (20) has 13 independent constants and takes the form

\[
[C_{ijkl}] = \begin{bmatrix}
C_{1111} & C_{1122} & C_{1133} & C_{1112} & 0 & 0 \\
C_{1122} & C_{2222} & C_{2233} & C_{2212} & 0 & 0 \\
C_{1133} & C_{2233} & C_{3333} & C_{3312} & 0 & 0 \\
C_{1112} & C_{2212} & C_{3312} & C_{1212} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{1313} & C_{1323} \\
0 & 0 & 0 & 0 & C_{1323} & C_{2323}
\end{bmatrix}. \] (27)

The orthotropic material has the symmetry of its elastic properties with respect to two orthogonal planes. Suppose that those planes are \( \hat{x}_1 - \hat{x}_2 \) and \( \hat{x}_2 - \hat{x}_3. \) Then, orthotropic material is the material that shows symmetry with respect to (22) and

\[ x_1^* = -x_1; \quad x_2^* = x_2; \quad x_3^* = x_3. \] (28)

The transformation matrix corresponding to (28) reads

\[
Q_2 = \begin{bmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}. \] (29)

When (29) is used in (3.2-7) and (23) we obtain
\[ C_{1112} = C_{2212} = C_{3312} = C_{1323} = 0, \] (30)

so that orthotropic material has 9 elastic constants.

The tetragonal material is an orthotropic material that has the same properties along two axes and different properties along the third axis. Thus, if elastic constants of an orthotropic material remain unchanged under the transformation

\[ x_1^* = x_1; \quad x_2^* = x_3; \quad x_3^* = -x_2, \] (31)
3.3 Anisotropic, orthotropic, and isotropic elastic body

described by

\[ Q_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \]

then this material is tetragonal. By using (32) in (3.2-7) and (23) we obtain

\[ C_{2222} = C_{3333}; \quad C_{1122} = C_{1133}; \quad C_{1212} = C_{1313}, \]

so that the tetragonal material has six independent elastic constants.

The cubic material is a tetragonal material that is invariant to an additional change of coordinates described by

\[ x'_1 = x_2; \quad x'_2 = -x_1; \quad x'_3 = x_3. \]

In matrix form, the transformation (34) corresponds to

\[ Q_4 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \]

Again by using (35) in (3.2-7) and (23) it follows that

\[ C_{1111} = C_{2222}; \quad C_{2233} = C_{1122}; \quad C_{2323} = C_{1212}. \]

Therefore cubic material has three independent constants and the matrix (27) in the case of cubic material has the form

\[ [C_{ijkl}] = \begin{bmatrix} C_{1111} & C_{1122} & C_{1122} & 0 & 0 & 0 \\ C_{1122} & C_{1111} & C_{1122} & 0 & 0 & 0 \\ C_{1122} & C_{1122} & C_{1111} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{1212} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{1212} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{1212} \end{bmatrix}. \]

Finally material is isotropic if it has the same elastic properties in all directions. It could be shown that a cubic material is isotropic if it is invariant under the transformation (see Bisplinghoff, Marr, and Pian (1965))

\[ x'_1 = \frac{\sqrt{3}}{2} x_1 + \frac{\sqrt{2}}{2} x_2; \quad x'_2 = -\frac{\sqrt{2}}{2} x_1 + \frac{\sqrt{2}}{2} x_2; \quad x'_3 = x_3, \]

or

\[ Q_5 = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \]

By using (39), (3.2-7), and (23) we obtain

\[ C_{1212} = \frac{C_{1111} - C_{1122}}{2}. \]
Thus, isotropic material has two independent elastic constants. The constants are usually denoted as

$$C_{122} = \lambda; \quad C_{111} = \lambda + 2\mu; \quad C_{1212} = \mu. \quad (41)$$

The constants $\lambda$ and $\mu$ are called Lamé constants. Note that, according to the simplified molecular theory of Poisson and Cauchy, isotropic material is described by one constant only. This is equivalent to the condition $\lambda = \mu$ in (41). Many experiments during the nineteenth century proved that, in general, $\lambda \neq \mu$ although for some special isotropic material it could be $\lambda = \mu$. We write (37) by using (41) in the form

$$[C_{ijkl}] = \begin{bmatrix} \lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda + 2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix}. \quad (42)$$

With (42) the generalized Hooke’s law (3.2-1) for isotropic material becomes

$$\begin{align*} 
\sigma_{11} &= \lambda \vartheta + 2\mu E_{11}; \\
\sigma_{33} &= \lambda \vartheta + 2\mu E_{33}; \\
\sigma_{13} &= 2\mu E_{13}; \\
\sigma_{i2} &= \lambda \vartheta + 2\mu E_{22}; \\
\sigma_{i1} &= \lambda \vartheta + 2\mu E_{12}; \\
\sigma_{23} &= 2\mu E_{23}, 
\end{align*}$$

where $\vartheta = \text{tr}E = E_{11} + E_{22} + E_{33}$. Equation (43) can be written as

$$\sigma_{ij} = \lambda \vartheta \delta_{ij} + 2\mu E_{ij}. \quad (44)$$

The generalized Hooke’s law for isotropic bodies (44) could be obtained directly from (3.2-1) and the condition that $C_{ijkl}$ is an isotropic fourth-order tensor. It is known that isotropic fourth-order tensors are linear combinations of the following tensors

$$\delta_{ij}\delta_{kl}; \quad \delta_{ik}\delta_{jl}; \quad \delta_{il}\delta_{jk}. \quad (45)$$

Therefore, in general, we have

$$C_{ijkl} = \lambda \delta_{ij}\delta_{kl} + \lambda_1 \delta_{ik}\delta_{jl} + \lambda_2 \delta_{il}\delta_{jk}, \quad (46)$$

where $\lambda$, $\lambda_1$, and $\lambda_2$ are constants. However, since $C_{ijkl}$ satisfies $C_{ijkl} = C_{ijlk} = C_{jikl}$ we must have $\lambda_1 = \lambda_2$. Let $\lambda_1 = \lambda_2 = \mu$. Then, (46) simplifies to

$$C_{ijkl} = \lambda \delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \quad (47)$$
By substituting (47) into (3.2-1) we obtain (44). Equation (44) could be easily solved for the components of deformation tensor $E_{ij}$. Namely, by taking $i = j$ in (44) it follows that

$$\text{tr } \sigma = \Theta = \sigma_{ii} = 3\lambda \vartheta + 2\mu E_{ii} = (3\lambda + 2\mu) \vartheta.$$  \hspace{1cm} (48)

From (48) we can determine the first invariant $\Theta$ of the stress tensor $\sigma$ in terms of the first invariant $\vartheta$ of the strain tensor $E$ as

$$\vartheta = \frac{\Theta}{3\lambda + 2\mu}.$$  \hspace{1cm} (49)

Also, from (44) it follows

$$E_{ij} = \frac{\sigma_{ij}}{2\mu} - \frac{\lambda}{2\mu} \delta_{ij},$$  \hspace{1cm} (50)

so that

$$E_{ij} = -\frac{\lambda \delta_{ij}}{2\mu(3\lambda + 2\mu)} \Theta + \frac{\sigma_{ij}}{2\mu}.$$  \hspace{1cm} (51)

The central result of this section is in equations (44) and (51). Both equations are based on the first term of series expansion of (3.1-3). In some applications the nonlinear relations between the strain and stress tensor is needed to properly describe the behavior of the material. One way to obtain such a relation (this is the so-called materially nonlinear theory) is to use second-order terms in the series expansion of (3.1-3); that is,

$$\sigma_{ij} = C_{ijkl} E_{kl} + \frac{1}{2} D_{ijklrs} E_{kl} E_{rs},$$  \hspace{1cm} (52)

where $D_{ijklrs}$ is a sixth-order tensor. It could be shown that there are 79 independent components of the tensor $D_{ijklrs}$. Therefore, in general, an elastic body described by (52) has 100 independent material constants. The basic drawback of the theory based on (52) is that it takes into account squares of the gradient of the displacement vector $(\partial u_i / \partial x_j)$ through the second term on the right-hand side of (52) and at the same time neglects the same order terms in the expression for the nonlinear strain tensor (since $E_{ij}$ is used in (52) and not $\tilde{E}_{ij}$ or $\tilde{E}_{ij}$).

One way to avoid this inconsistency is to use the stress-strain relation in the form (44) with a nonlinear strain tensor instead of a linear one. Then, with $\tilde{E}_{ij}$ instead of $E_{ij}$ we obtain (see Lurie (1980, p. 151))

$$\sigma_{ij} = \lambda (\text{tr } \tilde{E}) \delta_{ij} + 2\mu \tilde{E}_{ij}.$$  \hspace{1cm} (53)

The material described by (53) is called the Seth material. Then, the consistent generalization of Hooke's law reads ($\tilde{E}_{ij}$ instead of $E_{ij}$ in (52))

$$\sigma_{ij} = C_{ijkl} \tilde{E}_{kl} + \frac{1}{2} D_{ijklrs} \tilde{E}_{kl} \tilde{E}_{rs}.$$  \hspace{1cm} (54)

Again, there are, in general, 100 independent material parameters in equation (54).
3.4 Lamé constants. Modulus of elasticity. Poisson ratio

In elasticity theory, instead of Lamé constants $\lambda$ and $\mu$ two other constants are often used: modulus of elasticity $E$ and Poisson’s ratio $\nu$. We now derive the connections between Lamé constants and “engineering” constants $E$ and $\nu$.

Consider a rod loaded as in Fig. 2 of Chapter 1. Suppose that the $x_1$ axis is oriented along the axis of the rod and that the other two axes are oriented so that $x_1, x_2,$ and $x_3$ form a rectangular Cartesian coordinate system. The stress state at any point of the rod is

$$
\sigma_{11} = \sigma = \frac{F}{A}; \quad \sigma_{22} = \sigma_{33} = \sigma_{12} = \sigma_{13} = \sigma_{23} = 0. 
$$

Since there are no body forces the stress components (1) satisfy the equilibrium equations (1.3-10). By using (1) in (3.3-51) we obtain

$$
E_{11} = \frac{(\lambda + \mu)\sigma}{\mu(3\lambda + 2\mu)}; \quad E_{22} = E_{33} = \frac{-\lambda\sigma}{2\mu(3\lambda + 2\mu)};
$$

$$
E_{12} = E_{13} = E_{23} = 0.
$$

From (2) it follows that the compatibility equations (2.7-14) are satisfied. Let us introduce new constants $E$ and $\nu$ by the following expressions

$$
\frac{E_{22}}{E_{11}} = \frac{-\lambda}{2(\lambda + \mu)} = -\nu; \quad E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu},
$$

so that (2) becomes

$$
E_{11} = \frac{\sigma}{E}; \quad E_{22} = E_{33} = -\frac{\sigma}{E} \nu = -\nu E_{11}.
$$

The quantity $E$ is called the modulus of elasticity or Young’s modulus and $\nu$ is called Poisson’s ratio. From (3) it follows that

$$
\lambda = \frac{E \nu}{(1 + \nu)(1 - 2\nu)}; \quad \mu = \frac{E}{2(1 + \nu)}.
$$

Note that the molecular theory of Poisson and Cauchy leads to $\lambda = \mu$, or

$$
\nu = \frac{1}{4}.
$$

The controversy about the number of independent constants needed to describe an isotropic elastic body was, actually, centered about the value of Poisson ratio. As we see from (6) Poisson and Cauchy molecular theory predict that only one constant is sufficient to describe isotropic elastic material. If $E$ is taken as this constant, then the ratio of longitudinal and
lateral strain (and $\nu$ is just this) is equal to $\nu = 1/4$. Experiments that were performed, especially by G. Wertheim in 1848, clearly indicated that $\nu \neq 1/4$ for some isotropic elastic materials so that two constants are needed in the relation connecting the stress and strain tensors.

Consider now an elastic body loaded by hydrostatic pressure on its boundary

$$p_n = -pn.$$  \hspace{1cm} (7)

The components of the stress tensor are

$$\sigma_{11} = \sigma_{22} = \sigma_{33} = -p; \quad \sigma_{12} = \sigma_{13} = \sigma_{23} = 0.$$ \hspace{1cm} (8)

In the present case the equilibrium equations are satisfied and from (3.3-51) follows

$$E_{11} = E_{22} = E_{33} = -\frac{p}{3\lambda + 2\mu}; \quad E_{12} = E_{13} = E_{23} = 0.$$ \hspace{1cm} (9)

Since $\theta = E_{ii}$ is equal to the cubical dilatation (see (2.9-15)) we obtain

$$e_v = \theta = E_{11} + E_{22} + E_{33} = -\frac{p}{\lambda + \frac{2}{3}\mu}.$$ \hspace{1cm} (10)

By introducing a constant $k$ called the bulk modulus by the expression

$$k = \lambda + \frac{2}{3}\mu,$$ \hspace{1cm} (11)

equation (10) becomes

$$e_v = \theta = \frac{p}{k}.$$ \hspace{1cm} (12)

By using (5) in (11) we obtain

$$k = \frac{E}{3(1 - 2\nu)}.$$ \hspace{1cm} (13)

Finally we use (5) in (3.3-44) and (3.3-51) to write Hooke's law and its inversion as

$$\sigma_{ij} = \frac{E}{(1 + \nu)(1 - 2\nu)}[(1 - 2\nu)E_{ij} + \nu \partial \delta_{ij}];$$

$$E_{ij} = \frac{1}{E}[(1 + \nu)\sigma_{ij} - \nu \Theta \delta_{ij}],$$ \hspace{1cm} (14)

where $\partial$ and $\Theta$ are the first invariants of the strain and stress tensors, respectively.

We state now inequalities imposed on $\lambda$ and $\mu$. Thermodynamic considerations (the strain energy density $U$ is positive) lead to the following restrictions on Lamé constants in the generalized Hooke's law (3.2-44)

$$k = \lambda + \frac{2}{3}\mu > 0; \quad \mu > 0.$$ \hspace{1cm} (15)
From (15), (3), and (5) it follows that

\[ E > 0; \quad \nu > -1. \quad (16) \]

Equation (13) can be used to obtain another bound on \( \nu \). Namely, when \( \nu = 1/2 \) we have \( k = \infty \). Material with such bulk modulus is called the incompressible elastic material since

\[ e_v = 0, \quad (17) \]

as follows from (12). Therefore, for linearly elastic material (14) we have

\[ -1 < \nu < \frac{1}{2}. \quad (18) \]

It is interesting that for some elastic materials the Poisson ratio \( \nu \) has a negative value (see Rothenburg, Berlin, and Bathurst (1991)). For such materials, extension of a rod in the axial direction causes expansion in the lateral direction.

For Seth material described by (3.3-53) we can derive, by the same procedure as used for (3.3-44) (see Lurie (1980))

\[ \bar{E}_{11} = \frac{\sigma_{11}}{E}; \quad \bar{E}_{22} = -\nu \bar{E}_{11}; \]

\[ 9\lambda + 5\mu > 0; \quad \mu > 0. \quad (19) \]

The restrictions (19) are stronger than (15) as is to be expected since (15) guarantees that \( U \) is positive for small deformations only.

### 3.5 Influence of temperature on the stress-strain relation

Our goal in this section is to present the constitutive equations for the elastic body subjected to a change of temperature. Those equations connect the stress tensor \( \sigma_{ij} \), strain tensor \( E_{ij} \), and the temperature \( T \). The constitutive equations of thermoelasticity could be derived from the Helmholtz free energy of the material defined by (see Müller (1994))

\[ H = U - sT, \quad (1) \]

where \( H = H(E_{ij}, s) \) is the free energy density, \( s \) is the entropy, and \( T \) is the absolute temperature. Assuming that the stress tensor \( \sigma \) and the entropy \( s \) do not depend, explicitly on the time derivatives of \( E_{ij} \) and \( T \) we obtain (see Nowacki (1975))

\[ \sigma_{ij} = \frac{\partial H}{\partial E_{ij}}; \quad s = -\frac{\partial H}{\partial T}. \quad (2) \]
Next we expand the function $H$ in a power series in the vicinity of the undeformed state ($E_{ij} = 0$) and for $T = T_0$. The result is

$$H(E_{ij}, T) = H(E_{ij} = 0, T = T_0) + \frac{\partial H(0, T_0)}{\partial E_{ij}} E_{ij} + \frac{\partial H(0, T_0)}{\partial T} (T - T_0)$$

$$+ \frac{1}{2} \left[ \frac{\partial^2 H(0, T_0)}{\partial E_{ij} \partial E_{kl}} E_{ij} E_{kl} + 2 \frac{\partial^2 H(0, T_0)}{\partial E_{ij} \partial T} E_{ij} (T - T_0) + \frac{\partial^2 H(0, T_0)}{\partial T^2} (T - T_0)^2 \right] + ...$$  \hspace{1cm} (3)

In the natural state the free energy $H$ and the entropy are equal to zero so that

$$H(E_{ij} = 0, T_0) = 0; \quad \frac{\partial H(H, T_0)}{\partial T} = 0. \hspace{1cm} (4)$$

By using (3) and (4) in (2) we obtain

$$\sigma_{ij} = \frac{\partial H(0, T_0)}{\partial E_{ij}} + \frac{\partial^2 H(0, T_0)}{\partial E_{ij} \partial E_{kl}} E_{kl} + \frac{\partial^2 H(0, T_0)}{\partial E_{ij} \partial T} (T - T_0),$$  \hspace{1cm} (5)

where we neglected the higher-order terms. Since in the natural state ($E_{ij} = 0, T = T_0$) the stress is equal to zero (i.e., $\sigma_{ij} = 0$) we must have

$$\frac{\partial H(0, T_0)}{\partial E_{ij}} = 0. \hspace{1cm} (6)$$

Therefore from (5) it follows that

$$\sigma_{ij} = c_{ijkl} E_{kl} - \beta_{ij} \ddot{\theta}, \hspace{1cm} (7)$$

where

$$\left( \frac{\partial \sigma_{ij}}{\partial E_{kl}} \right)_T = c_{ijkl}; \quad \left( \frac{\partial \sigma_{ij}}{\partial T} \right)_{E_{ij}} = -\beta_{ij}; \quad \ddot{\theta} = T - T_0. \hspace{1cm} (8)$$

In writing (8) we indicated which variable is kept constant when taking partial derivatives. Equation (7) represents the Duhamel-Neumann constitutive equation for the linear anisotropic thermoelastic body. The constants $c_{ijkl}$ depend on the material of the body and are called elasticities in the isothermal state. The constants $\beta_{ij}$ are material constants describing mechanical and thermal properties of the material. In the case when the body is isotropic, by arguments similar to those presented in Section 3.3, equation (7) could be reduced to

$$\sigma_{ij} = [\lambda \dot{\theta} - \gamma \ddot{\theta}] \delta_{ij} + 2 \mu E_{ij}. \hspace{1cm} (9)$$

In (9) we used $\lambda$ and $\mu$ to denote the Lamé constants and $\gamma$ to denote the material constant that, as we shall show, is connected with the thermal
expansion coefficient. Note that (9) could be postulated as a constitutive equation. By using the same procedure as in Section 3.3 we could solve (9) for $E_{ij}$ to obtain

$$\dot{\theta} = E_{ii} = \frac{\gamma}{k} \ddot{\theta} + \frac{\Theta}{3\lambda + 2\mu},$$  \hspace{1cm} (10)

where $k$ is given by (3.4-15), $\Theta = \sigma_{ii} = \sigma_{11} + \sigma_{22} + \sigma_{33}$, and (compare with (3.3-49) and (3.3-51))

$$E_{ij} = \frac{-\lambda \delta_{ij}}{2\mu(3\lambda + 2\mu)} \Theta + \frac{\sigma_{ij}}{2\mu} + \frac{\gamma}{3k} \dot{\theta} \delta_{ij}. \hspace{1cm} (11)$$

To determine the physical meaning of $\gamma$ we consider the free thermal expansion of the body, that is, the case when $\sigma_{ij} = 0$ and $\dot{\theta} \neq 0$. Then, from (11) we obtain

$$E_{ij}^\theta = \frac{\gamma}{3k} \dot{\theta} \delta_{ij}. \hspace{1cm} (12)$$

If we use $\alpha_t$ to denote the linear thermal expansion coefficient, then from (12) it follows that

$$\alpha_t = \frac{\gamma}{3\lambda + 2\mu}. \hspace{1cm} (13)$$

Therefore, $\gamma$ is connected with the linear thermal expansion coefficient $\alpha_t$ through (13).

In solving the problems of thermoelasticity the field of temperature difference $\overline{\theta}$ in most cases is not known in advance. It must be determined from the generalized heat conduction equation. To derive this equation we need to consider the motion of a body. We start by defining the velocity and acceleration.

If a body moves, the position vector of an arbitrary point $r$ becomes a function of the position vector $R$ of the same point in the reference configuration and the time $t$. Thus instead of $r = r(X_i)$, as in Section 2.2, we have

$$r = r(X_i, t). \hspace{1cm} (14)$$

If the body is in the reference configuration $\kappa_0$ at the time instant $t_0$, then $r$ must satisfy $r(X_i, t_0) = R(X_i)$. The velocity and acceleration vector are

$$\mathbf{v} = v_i \mathbf{e}_i = \frac{\partial r}{\partial t} = \frac{\partial x_i}{\partial t} \mathbf{e}_i = \frac{\partial u_i}{\partial t} \mathbf{e}_i, \hspace{1cm} (15)$$

and

$$\mathbf{a} = \frac{\partial \mathbf{v}}{\partial t} = \frac{\partial^2 r}{\partial t^2} = \frac{\partial^2 x_i}{\partial t^2} \mathbf{e}_i = \frac{\partial^2 u_i}{\partial t^2} \mathbf{e}_i, \hspace{1cm} (16)$$

respectively. The generalized heat conduction equation is derived from the first and second law of thermodynamics (see Nowacki (1975) and the ref-
3.5 Influence of temperature on the stress-strain relation

In its final form, for isotropic bodies, it reads\(^5\)

\[
\frac{\partial^2 \tilde{\theta}}{\partial X_1^2} + \frac{\partial^2 \tilde{\theta}}{\partial X_2^2} + \frac{\partial^2 \tilde{\theta}}{\partial X_3^2} - \frac{\eta}{\lambda_0} \frac{\partial \tilde{\theta}}{\partial t} \left(1 + \frac{\tilde{\theta}}{T_0}\right) + \frac{q}{\lambda_0} = \frac{1}{\kappa} \frac{\partial \tilde{\theta}}{\partial t}.
\]

(17)

In (17) we use \(q\) to denote the rate of heat generated in a unit volume of the body, \(\lambda_0\) is the thermal conductivity. The time derivative of the first invariant of the strain tensor \(\theta\) is

\[
\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial t} [E_{11} + E_{22} + E_{33}] = \frac{\partial}{\partial t} \text{div } \mathbf{u} = \frac{\partial}{\partial t} \left( \frac{\partial u_1}{\partial X_1} + \frac{\partial u_2}{\partial X_2} + \frac{\partial u_3}{\partial X_3} \right).
\]

(18)

Also the parameters \(\kappa\) (thermal diffusivity) and \(\eta\) are given as

\[
\kappa = \frac{\lambda_0}{c_\varepsilon}; \quad \eta = \frac{\gamma T_0}{\lambda_0},
\]

(19)

where \(c_\varepsilon\) is the specific heat at constant strain.

It is obvious from (17) there is coupling between the heat conduction equation and the motion of the body. Therefore we have to adjoin to (17) equations of motion. They could be derived from equilibrium equations (1.3-4). Namely, by using D’Alembert’s principle we add the inertial forces to the distributed forces field. The inertial forces in a volume element \(dv\) are

\[
f_1^{in} = -\rho_0 \frac{\partial^2 u_1}{\partial t^2} dv; \quad f_2^{in} = -\rho_0 \frac{\partial^2 u_2}{\partial t^2} dv; \quad f_3^{in} = -\rho_0 \frac{\partial^2 u_3}{\partial t^2} dv,
\]

(20)

where \(\rho_0\) is the mass density in the deformed configuration, which in the linear approximation that we use is equal to the mass density in the initial (undeformed) configuration. By using (20) in (1.3-4) we obtain the equations of motion in the form

\[
\begin{align*}
\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} + \frac{\partial \sigma_{31}}{\partial x_3} + f_1 &= \rho_0 \frac{\partial^2 u_1}{\partial t^2}; \\
\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{32}}{\partial x_3} + f_2 &= \rho_0 \frac{\partial^2 u_2}{\partial t^2}; \\
\frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + f_3 &= \rho_0 \frac{\partial^2 u_3}{\partial t^2}.
\end{align*}
\]

(21)

Note that for the system (9),(17),(21), and (2.2-49) the initial and boundary conditions must be specified.

\(^5\) Basically the idea of including an additional term arising from deformation in the (standard) heat conduction equation originated with Russian physicist N. A. Umov who stated this idea in his 1871 book.
3.6 Hooke’s law in cylindrical and spherical coordinate systems

Hooke’s law (3.3-44) could be written in other coordinate systems by using the tensor transformation laws. Since we already have the strain tensor written in cylindrical and spherical coordinate systems (see Section 2.6) we can use these expressions to obtain the following results.

1. Cylindrical coordinate system

\[
\vartheta = E_{rr} + E_{\theta \theta} + E_{zz} = \frac{u_r}{r} + \frac{\partial u_r}{\partial r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z};
\]
\[
\sigma_{rr} = \lambda \vartheta + 2\mu \frac{\partial u_r}{\partial r}; \quad \sigma_{\theta \theta} = \lambda \vartheta + 2\mu \frac{1}{r} \left( \frac{\partial u_\theta}{\partial \theta} + u_r \right);
\]
\[
\sigma_{zz} = \lambda \vartheta + 2\mu \frac{\partial u_z}{\partial z}; \quad \sigma_{r \theta} = \mu \left[ \frac{1}{r} \left( \frac{\partial u_r}{\partial \theta} - u_\theta \right) + \frac{\partial u_\theta}{\partial r} \right];
\]
\[
\sigma_{r z} = \mu \left( \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right); \quad \sigma_{z \theta} = \mu \left( \frac{1}{r} \frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z} \right). \tag{1}
\]

In writing (1) we used the linear part of (2.6-6).

2. Spherical coordinate system

\[
\vartheta = 2 \frac{u_\rho}{\rho} + \frac{\partial u_\rho}{\partial \rho} + \frac{1}{\rho} \frac{\partial u_\phi}{\partial \phi} + \frac{1}{\rho \sin \varphi} \frac{\partial u_\theta}{\partial \theta} + \frac{u_\phi}{\rho} \cot \varphi;
\]
\[
\sigma_{\rho \rho} = \lambda \vartheta + 2\mu \frac{\partial u_\rho}{\partial \rho}; \quad \sigma_{\phi \phi} = \lambda \vartheta + 2\mu \left( \frac{1}{\rho} \frac{\partial u_\phi}{\partial \phi} + \frac{u_\phi}{\rho} \right);
\]
\[
\sigma_{\theta \theta} = \lambda \vartheta + 2\mu \left( \frac{1}{\rho \sin \varphi} \frac{\partial u_\theta}{\partial \theta} + \frac{u_\rho}{\rho} + \frac{u_\phi}{\rho \cot \varphi} \right);
\]
\[
\sigma_{\phi \rho} = \mu \left[ \frac{1}{\rho} \left( \frac{\partial u_\rho}{\partial \phi} - u_\phi \right) + \frac{\partial u_\phi}{\partial \rho} \right];
\]
\[
\sigma_{\phi \theta} = \mu \left( \frac{1}{\rho \sin \varphi} \frac{\partial u_\phi}{\partial \theta} - \cot \varphi \frac{u_\theta}{\rho} + \frac{1}{\rho} \frac{\partial u_\theta}{\partial \phi} \right);
\]
\[
\sigma_{\theta \rho} = \mu \left( \frac{\partial u_\theta}{\partial \rho} + \frac{1}{\rho \sin \varphi} \frac{\partial u_\rho}{\partial \theta} - \frac{u_\theta}{\rho} \right). \tag{2}
\]

In writing (2) we used the linear part of (2.6-11).

For the case of a thermoelastic body the stress-strain relations in a cylindrical and spherical coordinate system could be obtained from (1) and (2) by replacing the term \((\lambda \vartheta)\) by \((\lambda \vartheta - \gamma \dot{\vartheta})\).
3.7 Beltrami-Michell compatibility conditions

In this section we present the form in which the compatibility equations (2.7-14) can be written for the linearly elastic body described by (3.3-44) or (3.4-14). First, we express the strain tensor \( E_{ij} \) in terms of the stress tensor \( \sigma_{ij} \). Then, from (3.4-14) we have, for example

\[
E_{22} = \frac{1}{E}[(1 - \nu)\sigma_{22} - \nu \Theta];
\]
\[
E_{33} = \frac{1}{E}[(1 - \nu)\sigma_{33} - \nu \Theta];
\]
\[
E_{23} = \frac{\sigma_{23}}{2\mu} = \frac{1 + \nu}{E} \sigma_{23}, \quad (1)
\]

where \( \Theta = \sigma_{11} + \sigma_{22} + \sigma_{33} \). Consider the system of compatibility equations (2.7-14). In what follows we replace the derivatives there with respect to \( X_i \) with the derivatives with respect to \( x_i \) (see (2.9-4)). Then, for example, equation (2.7-14) is

\[
\frac{\partial^2 E_{22}}{\partial x_3 \partial x_3} + \frac{\partial^2 E_{33}}{\partial x_2 \partial x_2} - 2 \frac{\partial^2 E_{23}}{\partial x_2 \partial x_3} = 0. \quad (2)
\]

By using (1) equation (2) becomes

\[
(1 + \nu) \left[ \frac{\partial^2 \sigma_{22}}{\partial x_3^2} + \frac{\partial^2 \sigma_{33}}{\partial x_2^2} \right] - \nu \left[ \frac{\partial^2 \Theta}{\partial x_3^2} + \frac{\partial^2 \Theta}{\partial x_2^2} \right] = 2(1 + \nu) \frac{\partial^2 \sigma_{23}}{\partial x_2 \partial x_3}. \quad (3)
\]

From the equilibrium equations (1.3-4), we have

\[
\frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + f_3 = 0;
\]
\[
\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} + f_2 = 0. \quad (4)
\]

By differentiating (4) with respect to \( x_3 \) and (4) with respect to \( x_2 \) and by adding the results, we obtain

\[
2 \frac{\partial^2 \sigma_{23}}{\partial x_2 \partial x_3} = -\frac{\partial^2 \sigma_{33}}{\partial x_3^2} - \frac{\partial^2 \sigma_{22}}{\partial x_2^2} - \frac{\partial}{\partial x_1} \left( \frac{\partial \sigma_{13}}{\partial x_3} + \frac{\partial \sigma_{12}}{\partial x_2} \right) - \frac{\partial f_3}{\partial x_3} - \frac{\partial f_2}{\partial x_2}. \quad (5)
\]

From the first of the equilibrium equations (1.3-4)

\[
\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} + f_1 = 0, \quad (6)
\]

we have

\[
\frac{\partial}{\partial x_1} \left( \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} \right) = \frac{\partial^2 \sigma_{11}}{\partial x_1^2} + \frac{\partial f_1}{\partial x_1}. \quad (7)
\]
By substituting (7) into (5) it follows that

\[ 2 \frac{\partial^2 \sigma_{23}}{\partial x_2 \partial x_3} = \frac{\partial^2 \sigma_{11}}{\partial x_1^2} - \frac{\partial^2 \sigma_{22}}{\partial x_2^2} - \frac{\partial^2 \sigma_{33}}{\partial x_3^2} + \frac{\partial f_1}{\partial x_1} - \frac{\partial f_2}{\partial x_2} - \frac{\partial f_3}{\partial x_3}. \]  

(8)

Using (8) in (3) we obtain

\[ (1 + \nu) \left( \frac{\partial^2 \sigma_{22}}{\partial x_2^2} + \frac{\partial^2 \sigma_{33}}{\partial x_3^2} \right) - \nu \left( \frac{\partial^2 \Theta}{\partial x_1^2} + \frac{\partial^2 \Theta}{\partial x_2^2} \right) = (1 + \nu) \left( \frac{\partial^2 \sigma_{11}}{\partial x_1^2} - \frac{\partial^2 \sigma_{22}}{\partial x_2^2} - \frac{\partial^2 \sigma_{33}}{\partial x_3^2} + \frac{\partial f_1}{\partial x_1} - \frac{\partial f_2}{\partial x_2} - \frac{\partial f_3}{\partial x_3} \right). \]  

(9)

Now, by combining terms equation (9) can be transformed into

\[ (1 + \nu) \left( \nabla^2 \Theta - \nabla^2 \sigma_{11} - \frac{\partial^2 \Theta}{\partial x_1^2} \right) - \nu \left( \nabla^2 \Theta - \frac{\partial^2 \Theta}{\partial x_1^2} \right) = (1 - \nu) \left( \frac{\partial f_1}{\partial x_1} - \frac{\partial f_2}{\partial x_2} - \frac{\partial f_3}{\partial x_3} \right). \]  

(10)

Similar analysis leads to two more equations of the form (10); that is,

\[ (1 - \nu) \left( \nabla^2 \Theta - \nabla^2 \sigma_{22} - \frac{\partial^2 \Theta}{\partial x_2^2} \right) - \nu \left( \nabla^2 \Theta - \frac{\partial^2 \Theta}{\partial x_2^2} \right) = (1 + \nu) \left( \frac{\partial f_2}{\partial x_2} - \frac{\partial f_1}{\partial x_1} - \frac{\partial f_3}{\partial x_3} \right); \]

\[ (1 - \nu) \left( \nabla^2 \Theta - \nabla^2 \sigma_{33} - \frac{\partial^2 \Theta}{\partial x_3^2} \right) - \nu \left( \nabla^2 \Theta - \frac{\partial^2 \Theta}{\partial x_3^2} \right) = (1 + \nu) \left( \frac{\partial f_3}{\partial x_3} - \frac{\partial f_1}{\partial x_1} - \frac{\partial f_2}{\partial x_2} \right). \]  

(11)

By adding (10) and (11) we obtain

\[ (1 - \nu) \nabla^2 \Theta = -(1 - \nu) \left( \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3} \right). \]  

(12)

Introducing the divergence of a vector field of body forces as

\[ \text{div } \mathbf{f} = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3}, \]  

(13)

equation (12) can be written as

\[ \nabla^2 \Theta = \frac{1 + \nu}{1 - \nu} \text{div } \mathbf{f}. \]  

(14)
Equation (14) has importance in the analysis that follows. Using (14) in (10) we finally obtain

$$\nabla^2 \sigma_{11} + \frac{1}{1 + \nu} \frac{\partial^2 \Theta}{\partial x_1^2} + \frac{\nu}{1 - \nu} \text{div} f + 2 \frac{\partial f_1}{\partial x_1} = 0. \quad (15)$$

By similar arguments we can derive two more equations of the form (15). Another group of compatibility equations can be obtained if we start from (2.7-14)\textsubscript{6}; for example,

$$\frac{\partial^2 E_{33}}{\partial x_1 \partial x_2} + \frac{\partial^2 E_{12}}{\partial x_3^2} - \frac{\partial^2 E_{23}}{\partial x_1 \partial x_3} + \frac{\partial^2 E_{31}}{\partial x_2 \partial x_3} = 0. \quad (16)$$

Then, by using (3.4-13) we have

$$E_{33} = \frac{1}{E} [(1 - \nu)\sigma_{33} - \nu \Theta]; \quad E_{23} = \frac{1 + \nu}{E} \sigma_{23}; \quad E_{31} = \frac{1 + \nu}{E} \sigma_{31}. \quad (17)$$

From (17) and (16) it follows that

$$(1 + \nu) \frac{\partial^2 \sigma_{33}}{\partial x_1 \partial x_2} - \nu \frac{\partial^2 \Theta}{\partial x_1 \partial x_2} = (1 + \nu) \left( - \frac{\partial^2 \sigma_{12}}{\partial x_3^2} - \frac{\partial^2 \sigma_{23}}{\partial x_1 \partial x_3} + \frac{\partial^2 \sigma_{31}}{\partial x_2 \partial x_3} \right). \quad (18)$$

By differentiating the equilibrium equations (6) and (4)\textsubscript{2} with respect to $x_2$ and $x_1$, respectively, and by adding the resulting expressions, we have

$$\frac{\partial^2 \sigma_{23}}{\partial x_1 \partial x_3} + \frac{\partial^2 \sigma_{31}}{\partial x_2 \partial x_3} = - \left( \frac{\partial^2 \sigma_{11}}{\partial x_1 \partial x_2} + \frac{\partial^2 \sigma_{22}}{\partial x_1 \partial x_2} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial f_1}{\partial x_2} + \frac{\partial f_2}{\partial x_2} \right). \quad (19)$$

When (19) is substituted in (18), the resulting expression is

$$\nabla^2 \sigma_{12} + \frac{1}{1 + \nu} \frac{\partial^2 \Theta}{\partial x_1 \partial x_2} + \frac{\partial f_1}{\partial x_2} + \frac{\partial f_2}{\partial x_1} = 0. \quad (20)$$

By similar arguments we can derive two more equations of the form (20). Then the complete set of compatibility equations (2.7-14) is transformed into

$$\nabla^2 \sigma_{ij} + \frac{1}{1 + \nu} \frac{\partial^2 \Theta}{\partial x_i \partial x_j} + \frac{\nu}{1 - \nu} (\text{div} f) \delta_{ij} + \frac{\partial f_i}{\partial x_j} + \frac{\partial f_j}{\partial x_i} = 0. \quad (21)$$

Equations (21) are called the Beltrami-Michell compatibility equations.
In Section 3.1 we took, as a motivation for the assumption (3.1-3), the experimentally obtained relation (3.1-2). There exists, however, an important difference between (3.1-2) and (3.1-3). Namely, in (3.1-3) the components of Cauchy’s stress tensor $\sigma_{ij}$ represent the force per unit area of the sections in the deformed configuration $\kappa$, whereas $\sigma_R$ in (3.1-2) is the force per unit area of the body in the undeformed configuration $\kappa_0$. Thus $\sigma_R$ is the Piola-Kirchhoff stress component. We show next the form of (3.1-2) if we take the generalized Hooke’s law in the form (3.3-53). Then, for the linear state of stress (axially loaded rod) we have

$$\sigma_{11} = \lambda(\text{tr} \bar{\mathbf{E}}) + 2\mu \bar{E}_{11},$$  \hspace{1cm} (1)

or (see (3.4-19))

$$\sigma = \sigma_{11} = E \bar{E}_{11},$$  \hspace{1cm} (2)

where $E = \mu(3\lambda + 2\mu)/(\lambda + \mu)$. From (2.2-47) we obtain

$$\varepsilon_{11} = e = \frac{1}{\sqrt{1 - 2\bar{E}_{11}}} - 1,$$  \hspace{1cm} (3)

so that

$$e = \frac{1}{\sqrt{1 - 2\sigma/E}} - 1.$$  \hspace{1cm} (4)

By solving (4) for $\sigma$ we obtain (see Parkus (1983))

$$\frac{\sigma}{E} = \frac{2e + e^2}{2(1 + e)^2}.$$  \hspace{1cm} (5)

From (5) it follows that: \textit{if the equation connecting stress and the (finite) strain tensor is assumed in the linear form (2) then the relation connecting stress and the relative elongation $e$ is a nonlinear function given by (5).}

To connect $\sigma_R$ with $e$ we start from the definition of $\sigma_R$ as

$$\sigma A = \sigma_R A_0,$$  \hspace{1cm} (6)

where $A$ and $A_0$ are the cross-sectional area of the rod in the deformed and undeformed configurations, respectively. By solving (6) for $\sigma_R$ we obtain

$$\sigma_R = \sigma \frac{A}{A_0}.$$  \hspace{1cm} (7)

Suppose that the rod has a square cross-section with the dimension $a_0$ in the undeformed state. In the deformed state the dimension of the cross-section is denoted by $a$ and can be calculated by using (3.4-19). The result

---

6This assumption is not essential. It could be proved that the ratio $A/A_0$ obtained here holds for arbitrary cross-sections.
3.8 Finite deformations in linear state of stress

is

\[ a = a_0 (1 + \varepsilon_{22}) = a_0 \left( \frac{1}{1 + 2\nu \sigma / E} \right)^{1/2}. \]  

(8)

Since \((A/A_0) = (a/a_0)^2\), we obtain

\[ \sigma_R = \sigma \frac{1}{1 + 2\nu \sigma / E}. \]  

(9)

Finally, from (4) and (9) it follows that

\[ \frac{\sigma_R}{E} = \frac{2e + e^2}{2[1 + (1 + \nu)(2e + e^2)]}. \]  

(10)

In Fig. 4 we show the dependence of \(\sigma_R\) on \(e\). As can be seen the dependence can be approximated by a linear function only for small values of \(e\). Thus we conclude that: **linear dependence of the stress tensor on the (finite) Euler-Almanasi strain tensor leads to the nonlinear dependence of Piola-Kirchhoff stress as a function of the relative elongation \(e\).**

![Figure 4](image-url)

That the functions \(\sigma_R - e\) and \(\sigma - e\) are nonlinear, at least for some materials, has been recognized for long time. Thus, instead of (3.1-1) there are in the literature (especially on rod theory) other nonlinear relations, such as (5) and (10), suitable for specific materials. Some of the possibilities (see Bell (1973, p. 115) and Labisch (1992)) are listed in the following:

- **linear law, Hooke**
  \[ e = a\sigma; \]

- **exponential law**
  \[ e = a\sigma^m; \]

- **parabolic law**
  \[ \sigma = ae - be^2; \]
hyperbolic law

\[ e = \frac{\sigma}{a - b\sigma}; \]
\[ e^2 = a\sigma^2 + b\sigma; \]

cubic and biquadric-parabolic law

\[ \sigma = ae + be^2 + ce^3; \]
\[ e = a\sigma + \beta\sigma^2 + \gamma\sigma^3; \]
\[ \sigma = ae^{be^2} + ce^3 + de^4; \]

exponential law

\[ \sigma = c\exp^{-1/e}; \]
\[ e = \exp m\sigma - 1; \]
\[ \sigma = c(\exp me - 1); \]
\[ e = \sigma(a + b\exp m\sigma); \]
\[ \sigma = \frac{e}{1 - e}\exp me; \]

British Royal Commission

\[ \sigma = Ae - Be^2; \]

Nelson

\[ e = a\sigma^n; \]

Stokes

\[ \sigma = \frac{ae}{1 + \beta e}; \]

Richardson

\[ \sigma = ae - be^2; \]

Bilello and Metzger

\[ \sigma = ae^{1/2}; \]

Labisch

\[ \sigma = a\frac{e^3}{1 + e}. \]

In all the formulas listed \( a, b, c, d, n, m, \alpha, \beta, \gamma, A, \) and \( B \) are constants that are to be determined experimentally.

The variety of proposed generalizations of Hooke's law was motivated by experimental results that could not be described by linear law. As a matter of fact, Bell (1973, p. 155) states: "The experiments of 280 years have demonstrated amply for every solid substance examined with sufficient care, that the strain resulting from small applied stress is not a linear function thereof."
1. Show that the principal directions of the stress tensor and strain tensor coincide in the case of an isotropic elastic body.

2. If at the given point of a linearly elastic body the normal stresses satisfy
\[ \frac{\sigma_{11}}{\sigma_{22}} = \frac{a}{b}, \quad \frac{\sigma_{22}}{\sigma_{33}} = \frac{b}{c}, \]
find \( E_{11}/E_{22} \) and \( E_{22}/E_{33} \).

3. Derive the connections between Lamé constants \( \lambda \) and \( \mu \) and the modulus of elasticity \( E \) and Poisson ratio \( \nu \), as shown in the next table. Therefore, to describe isotropic elastic material any two of four constants (\( \lambda \), \( \mu \), \( E \), and \( \nu \)) can be used.

<table>
<thead>
<tr>
<th>( \lambda, \mu )</th>
<th>( \lambda )</th>
<th>( \mu )</th>
<th>( E )</th>
<th>( \nu )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda, \nu )</td>
<td>( \lambda \mu(1-2\nu) )</td>
<td>( 2\nu )</td>
<td>( \frac{\nu(3\lambda+2\mu)}{\lambda+\mu} )</td>
<td>( \lambda )</td>
</tr>
<tr>
<td>( \mu, E )</td>
<td>( \frac{\mu(E-2\nu)}{3\mu-E} )</td>
<td>( \frac{\mu(1-2\nu)}{2\nu} )</td>
<td>( \frac{E}{\nu} )</td>
<td>( \frac{E}{2\nu} )</td>
</tr>
<tr>
<td>( E, \nu )</td>
<td>( \frac{(1+\nu)(1-2\nu)}{2(1+\nu)} )</td>
<td>( \frac{E}{2(1+\nu)} )</td>
<td>( \frac{E}{(1+\nu)(1-2\nu)} )</td>
<td>( \frac{E}{2(1+\nu)} )</td>
</tr>
</tbody>
</table>

4. Show that the stress field
\[ \sigma_{11} = c[x_2^2 + v(x_1^2 - x_2^2)]; \quad \sigma_{22} = c[x_1^2 + v(x_2^2 - x_1^2)]; \]
\[ \sigma_{33} = cv(x_1^2 + x_2^2); \quad \sigma_{12} = -2cvx_1x_2; \quad \sigma_{13} = \sigma_{23} = 0, \]
where \( c \) is a constant, satisfies the equilibrium equations with zero body forces. Are the compatibility equations satisfied?

5. A straight rod is loaded along its axis that coincides with the \( \bar{x}_3 \) axis of a rectangular Cartesian coordinate system \( \bar{x}_1 - \bar{x}_2 - \bar{x}_3 \). The stress tensor component \( \sigma_{33} = \sigma \) could be controlled. The lateral surface of the rod is fixed in such a way that \( E_{22} = E_{33} = 0 \). Determine the “apparent” modulus of elasticity defined as
\[ E_{\text{apparent}} = \frac{\sigma}{E_{33}}. \]

6. Consider a cube made of material described by (3.3-53). Suppose that the cube is loaded so that the normal stresses on its sides are \( \sigma_1 = \sigma \), \( \sigma_2 = \sigma_3 = 0 \). Show that the cubical dilatation is given as
\[ e_\nu = 1 - \left(1 + \frac{2\nu\sigma}{E}\right) \sqrt{1 - \frac{2\sigma}{E}}, \]
where \( E \) is the modulus of elasticity and \( \nu \) Poisson ratio. Show further that \( \nu = 1/2 \) guarantees that \( e_\nu = 0 \) only if the ratio \( \sigma/E \) is small.
7. Show that the stress vector $\mathbf{P}_n$ could be expressed for the isotropic body described by (3.3-44) as (Love 1944, p. 134)

$$\mathbf{P}_n = \mathbf{P}_n(\mathbf{u}) = 2\mu \frac{\partial \mathbf{u}}{\partial \mathbf{n}} + \lambda \mathbf{n} \cdot \text{div} \mathbf{u} + \mu (\mathbf{n} \times \text{curl} \mathbf{n})$$

where $\frac{\partial (\cdot)}{\partial \mathbf{n}}$ is the directional derivative of $(\cdot)$, that is,

$$\frac{\partial \mathbf{u}}{\partial \mathbf{n}} = \frac{\partial u_i}{\partial x_j} n_j \mathbf{e}_i.$$

Also $\times$ denotes the cross product and $\text{div}$ and $\text{curl}$ of a vector field are defined as

$$\text{div} \mathbf{u} = \frac{\partial u_i}{\partial x_i};$$

$$\text{curl} \mathbf{u} = \left( \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} \right) \mathbf{e}_1 + \left( \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1} \right) \mathbf{e}_2 + \left( \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right) \mathbf{e}_3.$$

The operator $\mathbf{P}_n$ is called the stress operator.

8. Determine $\mathbf{P}_n(\mathbf{u})$ in cylindrical and spherical coordinate systems. First determine $\text{curl} \mathbf{u}$ in cylindrical and spherical coordinate systems as follows.

a) For cylindrical coordinate system $(r, \theta, z)$ start from the relations $x_1 = r \cos \theta; x_2 = r \sin \theta; x_3 = z$, to obtain

$$dx_1 = \cos \theta dr - r \sin \theta d\theta;$$
$$dx_2 = \sin \theta dr + r \cos \theta d\theta;$$
$$dx_3 = dz.$$  \hspace{1cm} (a)

b) Solve $(a)_{1,2}$ for $dr$ and $d\theta$, to obtain

$$dr = \cos \theta dx_1 + \sin \theta dx_2; \quad d\theta = -\frac{\sin \theta}{r} dx_1 + \frac{\cos \theta}{r} dx_2,$$

so that

$$\frac{\partial r}{\partial x_1} = \cos \theta; \quad \frac{\partial r}{\partial x_2} = \sin \theta; \quad \frac{\partial \theta}{\partial x_1} = -\frac{\sin \theta}{r}; \quad \frac{\partial \theta}{\partial x_2} = \frac{\cos \theta}{r}. \hspace{1cm} (b)$$

Then calculate

$$\frac{\partial u_2}{\partial x_1} = \frac{\partial u_2}{\partial r} \frac{\partial r}{\partial x_1} + \frac{\partial u_2}{\partial \theta} \frac{\partial \theta}{\partial x_1} = \frac{\partial u_2}{\partial r} \cos \theta - \frac{\partial u_2}{\partial \theta} \frac{\sin \theta}{r}. \hspace{1cm} (c)$$
c) Use (c) and similar expressions together with \( \mathbf{e}_r = \cos \theta \mathbf{e}_1 + \sin \theta \mathbf{e}_2; \mathbf{e}_\theta = -\sin \theta \mathbf{e}_1 + \cos \theta \mathbf{e}_2; \mathbf{e}_z = \mathbf{e}_3 \) (see Section 1.5) in the last equation of Problem 7 to obtain

\[
curl \mathbf{u} = \frac{1}{r} \left[ \frac{\partial u_z}{\partial \theta} - r \frac{\partial u_\theta}{\partial z} \right] \mathbf{e}_r + \left[ \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \right] \mathbf{e}_\theta \\
+ \frac{1}{r} \left[ \frac{\partial}{\partial r} (ru_\theta) - \frac{\partial u_r}{\partial \theta} \right] \mathbf{e}_z.
\]

(d) By similar analysis, show that in the spherical system \((\rho, \varphi, \theta)\) we have

\[
curl \mathbf{u} = \frac{1}{\rho \sin \varphi} \left[ \frac{\partial}{\partial \varphi} (u_\theta \sin \varphi) - \frac{\partial u_\varphi}{\partial \theta} \right] \mathbf{e}_\rho \\
+ \frac{1}{\rho} \left[ \frac{\partial}{\partial \rho} (\rho u_\varphi) - \frac{\partial u_\rho}{\partial \varphi} \right] \mathbf{e}_\theta \\
+ \frac{1}{\rho \sin \varphi} \left[ \frac{\partial u_\rho}{\partial \theta} - \sin \varphi \frac{\partial}{\partial \rho} (\rho u_\theta) \right] \mathbf{e}_\varphi.
\]

9. By using (2.4-17), that is, \( \mathbf{w} = \frac{1}{2} \) curl \( \mathbf{u} = \frac{1}{2} \nabla \times \mathbf{u} \), determine the components of the vector \( \mathbf{w} \) in cylindrical and spherical coordinate systems.
Chapter 4

Boundary Value Problems of Elasticity Theory

4.1 Introduction

In this section we present the basic systems of equations of elasticity theory. The systems consist of equilibrium equations, constitutive equations (Hooke’s law or Duhamel-Neumann law), and compatibility equations. To these sets of equations the body forces and boundary and initial conditions must be added.

To solve a problem in elasticity theory one usually has to face a serious mathematical task. The first question to be answered, before the solution procedure actually starts, is the question of existence of a solution in a given functional space. Depending on the space to which the solution belongs, we distinguish the classical and generalized (or weak) solutions of elasticity problems. We refer to the book of Fichera (1972) for the classical results on the existence theorems in elasticity theory. Another important question is the uniqueness of solution. For recent results in this direction, see Knops and Payne (1971). We present the Kirchhoff proof of the uniqueness of solution only. We distinguish equations of equilibrium and equations of motion for linear elasticity and linear thermoelasticity. Since we are considering small deformations and small displacements, in all equations that follow we use $X_i = x_i$ (see (2.9-4)). The basic systems of equations are:

1. Elasticity

Three equations of motion (3.5-21) are in the form

$$\frac{\partial \sigma_{ij}}{\partial x_j} + f_i = \rho_0 \frac{\partial^2 u_i}{\partial t^2}. \tag{1}$$

Six equations of Hooke’s law that for an isotropic body are given by (3.3-44),

$$\sigma_{ij} = \lambda \partial \delta_{ij} + 2 \mu E_{ij}, \tag{2}$$

where

$$E_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right). \tag{3}$$

To the system (1), (2) we adjoin the compatibility equations (2.7-13) in the form

$$\frac{\partial^2 E_{im}}{\partial x_j \partial x_l} + \frac{\partial^2 E_{jl}}{\partial x_i \partial x_m} - \frac{\partial^2 E_{il}}{\partial x_j \partial x_m} - \frac{\partial^2 E_{jm}}{\partial x_i \partial x_l} = 0. \tag{4}$$
Thus, fifteen dependent variables \( \sigma_{ij}, u_i, E_{ij} \) have to satisfy fifteen equations (1), (2), (3), and in addition system (4). From (2) we have

\[
\frac{\partial \sigma_{ij}}{\partial x_j} = \lambda \delta_{ij} \frac{\partial \vartheta}{\partial x_j} + \mu \frac{\partial}{\partial x_j} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right). \tag{5}
\]

By substituting (5) into (1) we obtain

\[
(\lambda + \mu) \frac{\partial \vartheta}{\partial x_j} + \mu \nabla^2 u_j + \left( f_j - \rho_0 \frac{\partial^2 u_j}{\partial t^2} \right) = 0. \tag{6}
\]

where \( \vartheta = E_{11} + E_{22} + E_{33} \) and

\[
\nabla^2 (\cdot) = \frac{\partial^2}{\partial x_1^2} (\cdot) + \frac{\partial^2}{\partial x_2^2} (\cdot) + \frac{\partial^2}{\partial x_3^2} (\cdot), \tag{7}
\]

is the Laplace operator.

Equations (6) are the Lamé differential equations. The solution of Lamé equations, say \( u_i = f_i(x_1, x_2, x_3, t) \), must satisfy the compatibility equations (4). In the case of equilibrium (6) becomes

\[
(\lambda + \mu) \frac{\partial \vartheta}{\partial x_j} + \mu \nabla^2 u_j + f_j = 0, \tag{8}
\]

or

\[
\frac{\partial \vartheta}{\partial x_j} + (1 - 2\nu) \nabla^2 u_j + \frac{2(1 + \nu)(1 - 2\nu)}{E} f_j = 0, \tag{9}
\]

where we used (3.4-5) to obtain

\[
\frac{\mu}{\lambda + \mu} = 1 - 2\nu; \quad \frac{1}{\lambda + \mu} = \frac{2(1 + \nu)(1 - 2\nu)}{E}. \tag{10}
\]

In the special case when the body forces are equal to zero, (8) reads

\[
(\lambda + \mu) \frac{\partial \vartheta}{\partial x_j} + \mu \nabla^2 u_j = 0. \tag{11}
\]

Suppose that the components of the body forces \( f_i \) are equal to zero, or constant. Then by differentiating (8) or (11) with respect to \( x_j \), \( j = 1, 2, 3 \) and summing the results, we obtain

\[
(\lambda + 2\mu) \nabla^2 \vartheta = 0, \tag{12}
\]

or

\[
\nabla^2 \vartheta = 0. \tag{13}
\]

From (13) we conclude that: if the body forces are absent or constant the first invariant of the strain tensor \( \vartheta = E_{11} + E_{22} + E_{33} \) is a harmonic function. Since \( \vartheta = e_v \) (see (2.9-15)) it follows that when the body forces
are absent or constant the cubical dilatation is a harmonic function. By applying Laplace’s operator to both sides of (11) we obtain

\[(\lambda + \mu) \frac{\partial}{\partial x_j} \nabla^2 \vartheta + \mu \nabla^2 \nabla^2 u_j = 0.\]  

(14)

By using (13) the condition (14) becomes

\[\nabla^2 \nabla^2 u_j = \nabla^4 u_j = 0,\]  

(15)

where

\[\nabla^4(\cdot) = \frac{\partial^4}{\partial x_1^4}(\cdot) + \frac{\partial^4}{\partial x_2^4}(\cdot) + \frac{\partial^4}{\partial x_3^4}(\cdot) + 2 \frac{\partial^4}{\partial x_1^2 \partial x_2^2}(\cdot) + 2 \frac{\partial^4}{\partial x_1^2 \partial x_3^2}(\cdot) + 2 \frac{\partial^4}{\partial x_2^2 \partial x_3^2}(\cdot),\]  

(16)

is the biharmonic operator. From (15) it follows that: if the body forces are absent or constant the components of the displacement vector \(u_j\) are biharmonic functions.

From Hooke’s law solved for the deformation tensor (3.3-51) or from (3.3-49) we have \(\vartheta = \Theta/(3\lambda + 2\mu)\). By using this and (13) we conclude that

\[\nabla^2 \Theta = 0,\]  

(17)

that is, if the body forces are absent or are constant the first invariant of the stress tensor is a harmonic function.

By applying the biharmonic operator to both sides of (3.3-51), we obtain

\[\nabla^4 E_{ij} = -\frac{\lambda \delta_{ij}}{2\mu(3\lambda + 2\mu)} \nabla^2 (\nabla^2 \Theta) + \frac{\nabla^4 \sigma_{ij}}{2\mu}.\]  

(18)

Since \(u_i\) satisfy (15) and \(\Theta\) satisfies (17) we conclude that

\[\nabla^4 \sigma_{ij} = 0;\]  

(19)

that is, if the body forces are absent or constant the stress components are biharmonic functions.

In some problems it is more convenient to use the Beltrami-Michell compatibility equations (3.7-21) instead of (4). Then we have to solve (1) and

\[\nabla^2 \sigma_{ij} + \frac{1}{1 + \nu} \frac{\partial^2 \Theta}{\partial x_i \partial x_j} + \frac{\nu}{1 - \nu} (\text{div} \ f) \delta_{ij} + \frac{\partial f_i}{\partial x_j} + \frac{\partial f_j}{\partial x_i} = 0,\]  

(20)

subject to specified boundary conditions.

2. Thermoelasticity

In the case of a thermoelastic body the basic equations are the three equations (1) and the constitutive equations (3.5-9)

\[\sigma_{ij} = [\lambda \vartheta - \gamma \bar{\vartheta}] \delta_{ij} + 2\mu E_{ij}.\]  

(21)
By using (3) in (21) and by substituting the result in (1) we obtain

\[
(\lambda + \mu) \frac{\partial \vartheta}{\partial x_j} + \mu \nabla^2 u_j + \rho \left( f_j - \frac{\partial^2 u_j}{\partial t^2} \right) - \gamma \frac{\partial \vartheta}{\partial x_j} = 0. \tag{22}
\]

Equation (22) must be solved together with the heat conduction equation (3.5-17)

\[
\frac{\partial^2 \vartheta}{\partial x_1^2} + \frac{\partial^2 \vartheta}{\partial x_2^2} + \frac{\partial^2 \vartheta}{\partial x_3^2} - \eta \frac{\partial \vartheta}{\partial t} \left( 1 + \frac{\vartheta}{T_0} \right) + \frac{q}{\lambda_0} = \frac{1}{\kappa} \frac{\partial \vartheta}{\partial t}. \tag{23}
\]

We simplify (23) in the usual way by assuming that \( |\vartheta/T_0| \ll 1 \). Also in what follows we use \( T \) to denote the temperature difference \( \vartheta \). Then (22) and (23) become

\[
(\lambda + \mu) \frac{\partial \vartheta}{\partial x_j} + \mu \nabla^2 u_j + \left( f_j - \rho_0 \frac{\partial^2 u_j}{\partial t^2} \right) - \gamma \frac{\partial T}{\partial x_j} = 0;
\]

\[
\frac{\partial^2 T}{\partial x_1^2} + \frac{\partial^2 T}{\partial x_2^2} + \frac{\partial^2 T}{\partial x_3^2} - \eta \frac{\partial T}{\partial t} + \frac{q}{\lambda_0} = \frac{1}{\kappa} \frac{\partial T}{\partial t}, \tag{24}
\]

where \( \vartheta = E_{11} + E_{22} + E_{33} = \partial u_1/\partial x_1 + \partial u_2/\partial x_2 + \partial u_3/\partial x_3 = \text{div}\, u \). To both (22) and (24) the boundary and initial conditions must be prescribed. Next we turn to the problem of specifying boundary conditions.

### 4.2 Classification of problems

As seen in the previous section, in an elasticity problem we have the following equations: three equations of motion; six constitutive equations (Hooke's law), and six strain displacement relations. From these fifteen equations the following fifteen unknowns must be determined: three component of the displacement vector \( u_i \); six components of the stress tensor \( \sigma_{ij} \), and six components of the strain tensor \( E_{ij} \). In the case of thermoelectricity there is one additional equation and one additional unknown, the temperature difference \( \vartheta \) (or \( T \)) and the heat conduction equation (4.1-24).

In practical application the most common type of loading and fixing of a body is to prescribe either the stress vector or the displacement vector on the boundary. Thus, in the case when the stress vector is prescribed on the boundary, we have (see (1.15-1))

\[
p_n = \sigma n = \hat{p}, \tag{1}
\]

where \( \hat{p} = \hat{p}(x_i) \) is the given function. If the displacement vector \( u \) is prescribed, then

\[
u = \hat{u}, \tag{2}
\]
where $\hat{\mathbf{u}} = \hat{\mathbf{u}}(x_i)$ is the given function.

For the systems (4.1-1) through (4.1-3) it is possible to formulate the following fundamental boundary value problems. We state those problems for the static case.

i) The first fundamental boundary value problem: given $f_i$ determine the six components of the stress tensor $\sigma_{ij}$ and the three components of the displacement vector $u_i$ inside the body if the following condition on the boundary $S$ of the body is prescribed\(^1\) (see Fig. 1a)

$$ (p_n)_i = \sigma_{ij} n_j = \hat{p}_i \quad \text{on } S, \quad (3) $$

where $\hat{p}_i$ is known function of the coordinates of a point on the boundary $S$ of the body $B$.

ii) The second fundamental boundary value problem: given $f_i$ determine the six components of the stress tensor $\sigma_{ij}$ and the three components of the displacement vector $u_i$ inside the body if the following condition on the boundary $S$ of the body is prescribed\(^2\) (see Fig. 1b)

$$ u_i(x_i) = \hat{u}_i \quad \text{on } S, \quad (4) $$

where $u_i$ is the known function of the coordinates of a point on the boundary $S$ of the body $B$.

![Figure 1](image)

iii) The third (mixed) fundamental boundary value problem: given $f_i$ determine the six components of the stress tensor $\sigma_{ij}$ and the three components of the displacement vector $u_i$ inside the body if the stress

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\(^1\) The boundary condition (3) should be interpreted as follows. Let $P \in B$ be an arbitrary point inside the body and $Q \in S$ be an arbitrary point on the surface of the body. Then $\lim_{P \to Q} p_n(P) = \hat{p}(Q)$.

\(^2\) Again (4) should be interpreted as $\lim_{P \to Q} u(P) = \hat{u}(Q)$ for $P \in B$ and $Q \in S$. 
vector is prescribed on the part of the boundary \( S_p \) and the displace­ment vector on the part of the boundary \( S_u \) \((S_p \cup S_u = S)\); that is

\[
(p_n)_i = \sigma_{ij} n_j = \hat{p}_i \quad \text{on} \quad S_p, \quad u_i(x_i) = \hat{u}_i \quad \text{on} \quad S_u, \tag{5}
\]

where \( \hat{p}_i \) and \( \hat{u}_i \) are the known functions of the coordinates of a point on the boundary of the body \( B \).

In the case of the first fundamental boundary value problem we have to solve the system (4.1-1) subject to (3). The stress components so determined are then used in (4.1-2) to find the strain components. Finally the displacement vector is determined from (4.1-3). It must satisfy the compatibility equation (4.1-4). Note that in accordance with (2.7-7) the displacement field is determined only up to the rigid body motion.

For the second fundamental boundary value problem we consider (4.1-6) subject to (4).

For the mixed boundary value problem we must solve (4.1-1),(4.1-2), (4.1-3),(4.1-6) subject to (3),(4).

The three types of boundary value problems formulated here are by no means the only boundary value problems that are important in the theory of elasticity. There are many other combinations of stress tensor and displacement vector components that have engineering significance. Some authors define those boundary value problems as: the fourth boundary value problem is the problem in which the normal component of the displacement vector \((u_n = u \cdot n)\) and the tangential component of the stress vector are prescribed on \( S \) or on part of \( S \); the fifth boundary value problem is the problem in which the normal component of the stress vector and the tangential component of the displacement vector \((u_t = u \cdot t)\) are prescribed; the sixth boundary value problem is the problem in which certain relations between components of the stress tensor and displacement vector are prescribed. The boundary value problems of these types have significance in contact problems, for example.

In the case of elastodynamics the same three types of fundamental problems may be formulated. The basic difference is that in the case of elastodynamics we must specify the initial values for the displacement vector as well as its time derivatives; that is, we must specify the initial value for the velocity vector. Thus, if we consider the motion of the body for time \( t \in [t_0, \infty) \), then we must specify

\[
\left. u_i(x_1, x_2, x_3, t) \right|_{t=t_0} = \hat{u}_i(x_1, x_2, x_3); \tag{6}
\]

\[
\left. \frac{\partial u_i}{\partial t}(x_1, x_2, x_3, t) \right|_{t=t_0} = \hat{v}_i(x_1, x_2, x_3).
\]
Finally in the case of thermoelasticity we have to prescribe the initial and boundary conditions for the generalized heat conduction equation (4.1-23). The initial temperature distribution may be described by a continuous or discontinuous function of the coordinates; that is,

$$T(x_1, x_2, x_3, t)|_{t=t_0} = \hat{T}(x_1, x_2, x_3),$$  \(7\)

where $\hat{T}(x_1, x_2, x_3)$ is a known function.

There may be a variety of conditions on the boundary of a thermoelastic body. Probably the simplest is when the temperature is prescribed over the whole boundary $S$ or part of it, denoted by $S_T$,

$$T(x_1, x_2, x_3, t) = \hat{T}(x_1, x_2, x_3, t) \quad \text{on} \quad S_T,$$  \(8\)

where $\hat{T}(x_1, x_2, x_3, t)$ is a known, possibly time-dependent, function and $S_T \subseteq S$.

On the boundary, or part of it, $S_q \subseteq S$, we can prescribe the heat flow. In this case we have

$$\frac{\partial T}{\partial n_i} = \hat{q}(x_1, x_2, x_3, t) \quad \text{on} \quad S_q,$$  \(9\)

where $\hat{q}(x_1, x_2, x_3, t)$ is a known functions. In the case of so-called Newton’s law of cooling, the boundary condition on it, $S_N \subseteq S$, reads

$$\frac{\partial T}{\partial n_i} = c_i(T^* - T) \quad \text{on} \quad S_N,$$  \(10\)

where $c_i$ are constants and $T^*$ denotes the temperature of the surrounding medium.

### 4.3 Lamé equations in coordinate systems

In Section 4.1 we derived the equations of motion in terms of the displacement vector (see (4.1-6)). These equations can be written in vector form as

$$\left(\lambda + \mu\right) \nabla \cdot \mathbf{u} + \mu \nabla^2 \mathbf{u} + \left( \mathbf{f} - \rho_0 \frac{\partial^2 \mathbf{u}}{\partial t^2} \right) = 0,$$  \(1\)

where $\nabla^2$ is defined by (4.1-7), $\nabla \cdot \mathbf{u} = \frac{\partial (\mathbf{u} \cdot \mathbf{e}_i)}{\partial x_i}$, and $\mathbf{w}$ is the rotation vector given by (2.4-14),(2.4-15). We now write (1) in the cylindrical coordinate system $(r, \theta, z)$. To do this we use (1.5-6), (3.6-1)
and the results of Problem 8, Chapter 3. Thus for the cylindrical coordinate system we have

\[
(\lambda + 2\mu)r \frac{\partial \vartheta}{\partial r} - 2\mu \left[ \frac{\partial \omega_z}{\partial \theta} - \frac{\partial}{\partial z} (r \omega_{\theta}) \right] + r \left[ f_r - \rho_0 \frac{\partial^2 u_r}{\partial t^2} \right] = 0;
\]

\[
(\lambda + 2\mu) \frac{1}{r} \frac{\partial \vartheta}{\partial \theta} - 2\mu \left[ \frac{\partial \omega_r}{\partial z} - \frac{\partial \omega_z}{\partial r} \right] + \left[ f_\theta - \rho_0 \frac{\partial^2 u_\theta}{\partial t^2} \right] = 0;
\]

\[
(\lambda + 2\mu) \frac{\partial \vartheta}{\partial z} - 2\mu \left[ \frac{\partial (r \omega_{\theta})}{\partial r} - \frac{\partial \omega_r}{\partial \theta} \right] + r \left[ f_z - \rho_0 \frac{\partial^2 u_z}{\partial t^2} \right] = 0, \tag{2}
\]

where

\[
\vartheta = E_{rr} + E_{\theta \theta} + E_{zz} = \frac{1}{r} \frac{\partial}{\partial r} (r u_r) + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z};
\]

\[
\omega_{\theta r} = \omega_z = \frac{1}{2r} \left[ \frac{\partial}{\partial r} (r u_\theta) - \frac{\partial u_r}{\partial \theta} \right];
\]

\[
\omega_{z \theta} = \omega_r = \frac{1}{2} \left[ \frac{\partial u_z}{\partial \theta} - r \frac{\partial u_\theta}{\partial z} \right];
\]

\[
\omega_{r z} = \omega_\theta = \frac{1}{2} \left[ \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \right]. \tag{3}
\]

Note that \( \omega_z, \omega_r, \) and \( \omega_\theta \) represent the components of the infinitesimal rotation tensor in the cylindrical coordinate system. In the special case of axially symmetric problems we have

\[
u_\theta = 0, \tag{4}
\]

and \( u_r \) and \( u_z \) independent of \( \theta \). Equations (2), (3) then become

\[
(\lambda + 2\mu)r \frac{\partial \vartheta}{\partial r} + 2\mu \frac{\partial}{\partial z} (r \omega_{\theta}) + r \left[ f_r - \rho_0 \frac{\partial^2 u_r}{\partial t^2} \right] = 0; \quad f_\theta = 0;
\]

\[
(\lambda + 2\mu) \frac{\partial \vartheta}{\partial z} - 2\mu \frac{\partial}{\partial r} (r \omega_{\theta}) + r \left[ f_z - \rho_0 \frac{\partial^2 u_z}{\partial t^2} \right] = 0, \tag{5}
\]

and

\[
\vartheta = E_{rr} + E_{\theta \theta} + E_{zz} = \frac{1}{r} \frac{\partial}{\partial r} (r u_r) + \frac{\partial u_z}{\partial z}. \tag{6}
\]

We consider next the form of (5) for static problems without body forces. Then, (5) takes the form

\[
\frac{\partial \vartheta}{\partial r} + \frac{2\mu}{(\lambda + 2\mu)} \frac{\partial \omega_{\theta}}{\partial z} = 0; \quad \frac{\partial \vartheta}{\partial z} - \frac{2\mu}{(\lambda + 2\mu) r} \frac{\partial}{\partial r} (r \omega_{\theta}) = 0. \tag{7}
\]

From (7) we can eliminate \( \vartheta \) to obtain

\[
\frac{\partial}{\partial r} \left[ \frac{1}{r} \frac{\partial}{\partial r} (r \omega_{\theta}) \right] + \frac{\partial^2 \omega_{\theta}}{\partial z^2} = 0. \tag{8}
\]
The solution procedure is now conducted as follows. First (8) is solved for \( \omega_\theta \). Then from (7) we determine the first invariant of the strain tensor \( \vartheta \). To determine the components of the displacement vector we use (3) and (6); that is,

\[
\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} = 2\omega_\theta; \quad \frac{1}{r} \frac{\partial}{\partial r} (ru_r) + \frac{\partial u_z}{\partial z} = \vartheta. \tag{9}
\]

In (9) the right-hand sides are known and we can solve for \( u_r \) and \( u_z \).

The system (9) could be reduced to a single partial differential equation by differentiating (9) with respect to \( z \) and (9) with respect to \( r \) and by adding the results. Then we obtain

\[
\frac{\partial}{\partial r} \left[ \frac{1}{r} \frac{\partial}{\partial r} (ru_r) \right] + \frac{\partial^2 u_r}{\partial z^2} = 2 \frac{\partial \omega_\theta}{\partial z} + \frac{\partial \vartheta}{\partial r}. \tag{10}
\]

We turn now to the spherical coordinate system \((\rho, \varphi, \theta)\). From (1.5-9) and (3.6-2) we have

\[
(\lambda + 2\mu) \rho \sin \varphi \frac{\partial \vartheta}{\partial \rho} - 2\mu \left[ \frac{\partial \omega_\varphi}{\partial \theta} - \frac{\partial}{\partial \varphi} (\omega_\theta \sin \varphi) \right] + \rho \sin \varphi \left( f_\rho - \rho_0 \frac{\partial^2 u_\rho}{\partial t^2} \right) = 0;
\]

\[
(\lambda + 2\mu) \frac{1}{\sin \varphi} \frac{\partial \vartheta}{\partial \theta} - 2\mu \left[ \frac{\partial \omega_\rho}{\partial \varphi} - \frac{\partial}{\partial \rho} (\rho \omega_\varphi) \right] + \rho \left( f_\varphi - \rho_0 \frac{\partial^2 u_\varphi}{\partial t^2} \right) = 0;
\]

\[
(\lambda + 2\mu) \sin \varphi \frac{\partial \vartheta}{\partial \varphi} - 2\mu \left[ \frac{\partial}{\partial \rho} (\rho \omega_\theta \sin \varphi) - \frac{\partial \omega_\rho}{\partial \theta} \right] + \rho \sin \varphi \left( f_\varphi - \rho_0 \frac{\partial^2 u_\varphi}{\partial t^2} \right) = 0, \tag{11}
\]

where

\[
\vartheta = E_{\rho \rho} + E_{\varphi \varphi} + E_{\theta \theta}
\]

\[
= \frac{1}{\rho^2} \frac{\partial}{\partial \rho} (\rho^2 u_\rho) + \frac{1}{\rho \sin \varphi} \left[ \frac{\partial}{\partial \varphi} (u_\varphi \sin \varphi) + \frac{\partial u_\theta}{\partial \theta} \right];
\]

\[
\omega_{\theta \varphi} = \omega_\rho = \frac{1}{2 \rho \sin \varphi} \left[ \frac{\partial u_\varphi}{\partial \theta} - \frac{\partial}{\partial \varphi} (u_\theta \sin \varphi) \right];
\]

\[
\omega_{\varphi \rho} = \omega_\theta = \frac{1}{2 \rho} \left[ \frac{\partial u_\rho}{\partial \varphi} - \frac{\partial}{\partial \rho} (\rho u_\varphi) \right];
\]

\[
\omega_{\rho \theta} = \omega_\varphi = \frac{1}{2 \rho \sin \varphi} \left[ \frac{\partial}{\partial \rho} (\rho u_\theta \sin \varphi) - \frac{\partial u_\rho}{\partial \theta} \right]. \tag{12}
\]
For axially symmetric problems in a spherical coordinate system we have all components of the displacement vector independent of $\theta$ and $u_\theta = 0$. Then (11) and (12) become

\[
(\lambda + 2\mu) \frac{\partial \vartheta}{\partial \rho} + 2\mu \frac{1}{\rho \sin \varphi} \frac{\partial}{\partial \varphi} (\omega_\theta \sin \varphi) + \rho \left( f_\rho - \rho_0 \frac{\partial^2 u_\rho}{\partial t^2} \right) = 0; \quad f_\theta = 0;
\]

\[
(\lambda + 2\mu) \frac{\partial \varphi}{\partial \varphi} - 2\mu \frac{1}{\sin \varphi} \frac{\partial}{\partial \rho} (\rho \omega_\theta \sin \varphi) + \rho \left( f_\varphi - \rho_0 \frac{\partial^2 u_\varphi}{\partial t^2} \right) = 0,
\]

(13)

and

\[
\vartheta = \frac{1}{\rho^2} \frac{\partial}{\partial \rho} (\rho^2 u_\rho) + \frac{1}{\rho \sin \varphi} \frac{\partial}{\partial \varphi} (u_\varphi \sin \varphi);
\]

\[
\omega_\theta = \frac{1}{2\rho} \left[ \frac{\partial u_\rho}{\partial \varphi} - \frac{\partial}{\partial \rho} (\rho u_\varphi) \right]; \quad \omega_\rho = \omega_\varphi = 0.
\]

(14)

We make two additional comments concerning the Lamé equations (1). They are obtained from Hooke’s law and the linear strain tensor. We may ask for the form of Lamé equations if the nonlinear strain tensor is used in Hooke’s law (3.3-53)\(^3\). It could be shown that the resulting nonlinear Lamé equations differ from the linear ones (1) by an additional term that has the form of a body force. For details of this analysis we refer to Klichevski (1963).

Another comment concerns thermoelasticity. Namely, (4.1-22) could be written in the form

\[
(\lambda + \mu) \nabla \text{div} \ u + \mu \nabla^2 u + \left( f - \rho_0 \frac{\partial^2 u}{\partial t^2} \right) - \gamma \nabla \vartheta
\]

\[
= (\lambda + 2\mu) \nabla \text{div} \ u - 2\mu (\text{curl} \ w) + \left( f - \rho_0 \frac{\partial^2 u}{\partial t^2} \right) - \gamma \nabla \vartheta = 0,
\]

(15)

where $\nabla \vartheta$ is the gradient of the temperature difference $\vartheta = T - T_0$. For cylindrical and spherical coordinate systems the gradients of the temperature difference are\(^4\)

\[
\nabla \vartheta = \nabla(T - T_0) = \frac{\partial(T - T_0)}{\partial r} e_r + \frac{1}{r} \frac{\partial(T - T_0)}{\partial \theta} e_\theta + \frac{\partial(T - T_0)}{\partial z} e_z,
\]

(16)

and

\[
\nabla \vartheta = \nabla(T - T_0) = \frac{\partial(T - T_0)}{\partial \rho} e_\rho + \frac{1}{\rho} \frac{\partial(T - T_0)}{\partial \varphi} e_\varphi + \frac{1}{\rho \sin \varphi} \frac{\partial(T - T_0)}{\partial \theta} e_\theta,
\]

(17)

\(^3\)The Lamé equations for the case when the nonlinear strain tensor is used were first obtained by N.A. Klichevski in 1940 (Klichevski 1963).

\(^4\)The expressions for the gradient of a scalar, divergence of a vector, and Laplace operator (divergence of a gradient) in cylindrical and spherical coordinate systems can be obtained from equations (b) and (c) of Problem 8 in Chapter 3.
respectively. By using (16), equations (2) in the case of thermoelasticity become

\[(\lambda + 2\mu)\frac{\partial \vartheta}{\partial r} - 2\mu \left[ \frac{\partial \omega_z}{\partial \theta} - \frac{\partial}{\partial z} (r \omega_\theta) \right] + \gamma \frac{r}{\frac{\partial \vartheta}{\partial r}} = 0;\]

\[(\lambda + 2\mu) \frac{1}{r} \frac{\partial \vartheta}{\partial \theta} - 2\mu \left[ \frac{\partial \omega_r}{\partial z} - \frac{\partial \omega_z}{\partial r} \right] + \left[ f_\theta - \rho_0 \frac{\partial^2 u_\theta}{\partial t^2} \right] - \gamma \frac{1}{r} \frac{\partial \vartheta}{\partial \theta} = 0;\]

\[(\lambda + 2\mu) r \frac{\partial \vartheta}{\partial z} - 2\mu \left[ \frac{\partial}{\partial r} (r \omega_\theta) - \frac{\partial \omega_r}{\partial \theta} \right] + \gamma \frac{r}{\frac{\partial \vartheta}{\partial z}} = 0,\]  

(18)
together with (3). In the spherical coordinate system we obtain

\[(\lambda + 2\mu) \rho \sin \varphi \frac{\partial \vartheta}{\partial \rho} - 2\mu \left[ \frac{\partial}{\partial \varphi} (\omega_\theta \sin \varphi) - \frac{\partial \omega_\varphi}{\partial \theta} \right] \]

\[+ \rho \sin \varphi \left[ f_\rho - \rho_0 \frac{\partial^2 u_\rho}{\partial t^2} \right] - \gamma \frac{1}{\sin \varphi} \frac{\partial \vartheta}{\partial \rho} = 0;\]

\[\left( \lambda + 2\mu \right) \frac{1}{\sin \varphi} \frac{\partial \vartheta}{\partial \theta} - 2\mu \left[ \frac{\partial \omega_\rho}{\partial \theta} - \frac{\partial}{\partial \rho} (\rho_\omega \sin \varphi) \right] \]

\[+ \rho \left( f_\theta - \rho_0 \frac{\partial^2 u_\theta}{\partial t^2} \right) - \gamma \frac{1}{\sin \varphi} \frac{\partial \vartheta}{\partial \rho} = 0;\]

\[(\lambda + 2\mu) \sin \varphi \frac{\partial \vartheta}{\partial \varphi} - 2\mu \left[ \frac{\partial \omega_\rho}{\partial \varphi} - \frac{\partial}{\partial \rho} (\rho_\omega \sin \varphi) \right] \]

\[+ \sin \varphi \left[ \rho \left( f_\varphi - \rho_0 \frac{\partial^2 u_\varphi}{\partial t^2} \right) - \gamma \frac{\partial \vartheta}{\partial \varphi} \right] = 0.\]  

(19)

To equations (18), (19) the heat conduction equation (4.1-23) must be added. In cylindrical coordinate system it reads

\[\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \vartheta}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \vartheta}{\partial \varphi^2} + \frac{\partial^2 \vartheta}{\partial z^2} - \eta \frac{\partial \vartheta}{\partial t} \left( 1 + \frac{\vartheta}{T_0} \right) + \frac{q}{\lambda_0} \frac{1}{\kappa} \frac{\partial \vartheta}{\partial \varphi};\]  

(20)

whereas in the spherical coordinate system we have

\[\frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left( \rho^2 \frac{\partial \vartheta}{\partial \rho} \right) + \frac{1}{\rho^2 \sin \varphi} \frac{\partial}{\partial \varphi} \left( \sin \varphi \frac{\partial \vartheta}{\partial \varphi} \right) \]

\[+ \frac{1}{\rho^2 \sin^2 \varphi} \frac{\partial^2 \vartheta}{\partial \varphi^2} - \eta \frac{\partial \vartheta}{\partial t} \left( 1 + \frac{\vartheta}{T_0} \right) + \frac{q}{\lambda_0} \frac{1}{\kappa} \frac{\partial \vartheta}{\partial \varphi}.\]  

(21)
4.4 Uniqueness of solution

In this section we assume that a solution to the fundamental boundary value problems defined in Section 4.2 exists. We prove that it is unique. We first treat the static problems.

Consider the integral

\[ J = \int_S \int (p_1 u_1 + p_2 u_2 + p_3 u_3) dS = \int_S \int p_i u_i dS, \]  

where \( p_i \) are components of the stress vector and \( S \) is the outer surface of the body \( B \). Since \( p_i = \sigma_{ij} n_j \), we have

\[ J = \int_S \int (P n_1 + Q n_2 + R n_3) dS, \]

where

\[ P = \sigma_{11} u_1 + \sigma_{21} u_2 + \sigma_{31} u_3; \quad Q = \sigma_{12} u_1 + \sigma_{22} u_2 + \sigma_{32} u_3; \quad R = \sigma_{13} u_1 + \sigma_{23} u_2 + \sigma_{33} u_3. \]

From the Gauss theorem, it follows that

\[ J = \int \int \int \left[ \frac{\partial P}{\partial x_1} + \frac{\partial Q}{\partial x_2} + \frac{\partial R}{\partial x_3} \right] dV, \]

where

\begin{align*}
\frac{\partial P}{\partial x_1} + \frac{\partial Q}{\partial x_2} + \frac{\partial R}{\partial x_3} &= u_1 \left( \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} \right) + u_2 \left( \frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} \right) \\
&\quad + u_3 \left( \frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} \right) + \sigma_{11} \frac{\partial u_1}{\partial x_1} + \sigma_{22} \frac{\partial u_2}{\partial x_2} + \sigma_{33} \frac{\partial u_3}{\partial x_3} \\
&\quad \quad + \sigma_{23} \left( \frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3} \right) + \sigma_{31} \left( \frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3} \right) + \sigma_{12} \left( \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right). \end{align*}

By using the equilibrium equations (1.3-4) we obtain

\[ \frac{\partial P}{\partial x_1} + \frac{\partial Q}{\partial x_2} + \frac{\partial R}{\partial x_3} = -(f_1 u_1 + f_2 u_2 + f_3 u_3) \]

\[ + [\sigma_{11} E_{11} + \sigma_{22} E_{22} + \sigma_{33} E_{33} + 2\sigma_{12} E_{12} + 2\sigma_{13} E_{13} + 2\sigma_{23} E_{23}]. \]
4.4 Uniqueness of solution

We show later (see (7.2-23)) that the expression in brackets in (6) is equal to $2u$ where $u$ is the strain energy density defined by (3.3-9). By using Hooke’s law (3.3-43) we can transform the expression in brackets to obtain

$$2u = [\sigma_{11}E_{11} + \sigma_{22}E_{22} + \sigma_{33}E_{33} + 2\sigma_{12}E_{12} + 2\sigma_{13}E_{13} + 2\sigma_{23}E_{23}]$$

$$= \lambda(E_{11} + E_{22} + E_{33})^2$$

$$+ 2\mu(E_{11}^2 + E_{22}^2 + E_{33}^2 + 2E_{12}^2 + 2E_{13}^2 + 2E_{23}^2).$$

(7)

Since $\mu > 0$, $\lambda > 0$ the function $u$ is a positive definite function of the strain tensor components and $u = 0$ if and only if all strain tensor components are equal to zero. By combining (1), (2), (4), and (6) we obtain

$$\int \int \int f_iu_i dV + \int \int p_iu_idS = 2\int \int \int udV.$$  

(8)

Equation (8) expresses Clapeyron’s theorem: if the body is in equilibrium under the action of body forces $f_i$ and surface forces $p_i$ then the work of these forces on the displacement $u_i$ from the unstressed state is the equal integral over the volume of the body of the function $2u$ given by (7).

We turn now to the proof of uniqueness. Suppose that any of the fundamental problems of Section 4.2 has two solutions. Let

$$u^I_i; \quad E^I_{ij}; \quad \sigma^I_{ij},$$

(9)

be the displacement strain and stress fields for the first and

$$u^{II}_i; \quad E^{II}_{ij}; \quad \sigma^{II}_{ij},$$

(10)

the corresponding fields for the second solution. Since the relevant equations that both solutions satisfy are linear, it follows that their difference

$$u_i = u_i^I - u_i^{II}; \quad E_{ij} = E_{ij}^I - E_{ij}^{II}; \quad \sigma_{ij} = \sigma_{ij}^I - \sigma_{ij}^{II},$$

(11)

is also the solution of equilibrium equations (1.3-4) with

$$f_1 = f_2 = f_3 = 0.$$  

(12)

Therefore from (8) we obtain

$$\int \int p_iu_idS = 2\int \int \int udV.$$  

(13)

Note that in the first fundamental boundary value problem the surface forces corresponding to $u_i$, $\sigma_{ij}$ are zero since both solutions satisfy the same boundary conditions. Therefore in the first fundamental problem

$$p_1 = p_2 = p_3 = 0, \quad \text{on } S.$$  

(14)
In the second fundamental problem we have

\[ u_1 = u_2 = u_3 = 0, \quad \text{on } S. \]  

(15)

Finally in the third fundamental problem we have

\[ \sigma_{ij}n_j = p_i = 0, \quad \text{on } S_p; \]

\[ u_i = 0, \quad \text{on } S_u. \]  

(16)

In all cases equation (13) implies

\[ \int \int \int u dV = 0. \]  

(17)

The function \( u \) is the positive definite function in the strain tensor components (see (7)). It then follows from (17) that \( E_{ij} = 0, \) or

\[ E^I_{ij} = E^{II}_{ij}. \]  

(18)

Since the strain tensors in both solutions are identical, it follows that the stresses are identical

\[ \sigma^I_{ij} = \sigma^{II}_{ij}. \]  

(19)

Thus we have the following uniqueness theorem: the solution of the fundamental static boundary value problems leads to the unique stress and strain fields.

The first proof of the uniqueness theorem is due to Kirchhoff (1859). Note that the relation (18) does not guarantee the uniqueness of displacements. Namely, from (18) it follows that two solutions \( u^I \) and \( u^{II} \) differ for the displacement field for which the strain tensor is equal to zero, that is, for displacement corresponding to the rigid body motion. In the second and third fundamental boundary value problems the displacement is determined uniquely since the displacement is prescribed over part of the boundary and rigid body displacements are eliminated.

The analysis presented here holds for simply and multiply connected bodies. However it does not hold for nonlinear elasticity theory (materially nonlinear (and) or geometrically nonlinear), given, for example, by (3.3-53). Buckling of elastic bodies is a consequence of nonuniqueness of solution of static problems of nonlinear elasticity (see Chapter 10).

Consider now the dynamic thermoelastic problem. In order to tackle it we have to connect mechanical and thermal properties of the system. Thus we integrate equation (4.1-1) over the volume of the body and use the divergence theorem to obtain

\[ \int \int \int u dV = 0. \]  

(17)

\[ \frac{1}{\partial} \left[ \partial \frac{\partial x_j}{\partial t} (\partial u_i/\partial t) + \partial \frac{\partial x_i}{\partial t} (\partial u_j/\partial t) \right] \]  

Only in the case of small deformation are components of \( d_{ij} \) equal to time derivatives of the strain tensor.
The two terms on the left-hand side of (20) represent the time rate of change of inner and kinetic energy. On the right-hand side the first term represents the power of the body forces and the second term represents the power of the surface forces (see Nowacki (1975 p. 40)). Any deformation process in a thermoelastic body has to satisfy (20), equations of motions (4.1-1)

\[
\frac{\partial \sigma_{ij}}{\partial x_i} + f_i = \rho_0 \frac{\partial^2 u_i}{\partial t^2},
\]

and the generalized heat conduction equation (4.1-24) that we write in the form (see (3.5-19))

\[
\lambda_0 \left( \frac{\partial^2 T}{\partial x_1^2} + \frac{\partial^2 T}{\partial x_2^2} + \frac{\partial^2 T}{\partial x_3^2} \right) - \gamma T_0 \frac{\partial \theta}{\partial t} + q = c_e \frac{\partial T}{\partial t},
\]

where \( c_e \) is the specific heat at constant strain and \( \gamma \) is given by (3.5-13).

To the system (21), (22) we adjoin the following boundary and initial conditions (see (4.2-5))

\[
(p_n)_i = \sigma_{ij} n_j = \hat{p}_i(x_i, t), \quad x_i \in S_p; \quad u_i(x_i) = \hat{u}_i(x_i, t), \quad x_i \in S_u;
\]

\[
\hat{q}_i n_i = \lambda_0 \frac{\partial T}{\partial x_i} n_i = f_q(x_i, t), \quad x_i \in S;
\]

\[
u_i(x_i, t = t_0) = \bar{u}_i(x_i), \quad \frac{\partial u_i(x_i, t = t_0)}{\partial t} = \bar{\nu}_i(x_i), \quad x_i \in V;
\]

\[
T(x_i, t = t_0) = f_T(x_i), \quad x_i \in V;
\]

where \( \hat{p}_i, \hat{u}_i, ..., f_T \) are known functions.

Suppose that there exist two solutions of the system (21), (22), (23). We denote these solutions by \( u_i^I, E_i^I, \sigma_i^I, T^I \) and \( u_i^{II}, E_i^{II}, \sigma_i^{II}, T^{II} \). The functions

\[
u_i = u_i^I - u_i^{II}; \quad E_i = E_i^I - E_i^{II}; \quad \sigma_i = \sigma_i^I - \sigma_i^{II}; \quad T = T^I - T^{II},
\]

satisfy the following system of equations

\[
\frac{\partial \sigma_{ij}}{\partial x_i} = \rho_0 \frac{\partial^2 u_i}{\partial t^2}; \quad \lambda_0 \left( \frac{\partial^2 T}{\partial x_1^2} + \frac{\partial^2 T}{\partial x_2^2} + \frac{\partial^2 T}{\partial x_3^2} \right) - \gamma T_0 \frac{\partial \theta}{\partial t} = c_e \frac{\partial T}{\partial t},
\]
subject to

\( (p_n)_i = \sigma_{ij} n_j = 0, \quad x_i \in S_p; \quad u_i(x_i) = 0, \quad x_i \in S_u; \)

\( \dot{q}_i n_i = \lambda_0 \frac{\partial T}{\partial x_i} n_i = 0, \quad x_i \in S; \)

\( u_i(x_i, t = t_0) = 0, \quad \frac{\partial u_i(x_i, t = t_0)}{\partial t} = 0, \quad T(x_i, t = t_0) = 0, \quad x_i \in V. \quad (26) \)

By using the first law of thermodynamics (20) for the solution (24) of (25), (26) we obtain

\[
\frac{\partial}{\partial t} \int \int \int \sigma_{ij} \frac{\partial E_{ij}}{\partial t} dV + \frac{1}{2} \frac{\partial}{\partial t} \int \int \rho_0 \left( \frac{\partial u_i}{\partial t} \right) \left( \frac{\partial u_i}{\partial t} \right) dV = 0. \quad (27)
\]

The stress tensor is given by (3.5-9) as

\[
\sigma_{ij} = \left[ \lambda \vartheta - \gamma T \right] \delta_{ij} + 2\mu E_{ij}
\]

\[
= 2\mu \left[ E_{ij} + \frac{\nu}{1 - 2\nu} \left( \vartheta - \frac{1 + \nu}{\nu} \alpha_t T \right) \delta_{ij} \right], \quad (28)
\]

where \( \vartheta = T \) (see Section 4.1) and where we have used that \( \lambda = 2\mu\nu/(1-2\nu) \) (see Problem 3, Chapter 3) and \( \gamma = \alpha_t (3\lambda + 2\mu) = 2\mu\alpha_t (1 + \nu)/(1 - 2\nu) \).

From equations (27), (28) it follows that

\[
2\mu \int \int \int \left[ E_{ij} \frac{\partial E_{ij}}{\partial t} + \frac{\nu}{1 - 2\nu} \vartheta \frac{\partial \vartheta}{\partial t} - \frac{1 + \nu}{1 - 2\nu} \alpha_t T \frac{\partial \vartheta}{\partial t} \right] dV
\]

\[
+ \frac{1}{2} \frac{\partial}{\partial t} \int \int \rho_0 \left( \frac{\partial u_i}{\partial t} \right) \left( \frac{\partial u_i}{\partial t} \right) dV = 0, \quad (29)
\]

or

\[
\frac{\partial}{\partial t} \left\{ \mu \int \int \int \left[ E_{ij} E_{ij} + \frac{\nu}{1 - 2\nu} \vartheta^2 \right] dV + \frac{1}{2} \int \int \rho_0 \left( \frac{\partial u_i}{\partial t} \right) \left( \frac{\partial u_i}{\partial t} \right) dV \right\}
\]

\[
= \frac{E\alpha_t}{1 - 2\nu} \int \int T \frac{\partial \vartheta}{\partial t} dV. \quad (30)
\]

We substitute now the term \( \partial \vartheta / \partial t \) from (25) into (30) to obtain

\[
\frac{\partial}{\partial t} \left\{ \mu \int \int \int \left[ E_{ij} E_{ij} + \frac{\nu}{1 - 2\nu} \vartheta^2 \right] dV + \frac{1}{2} \int \int \rho_0 \left( \frac{\partial u_i}{\partial t} \right) \left( \frac{\partial u_i}{\partial t} \right) dV \right\}
\]

\[
= \frac{E\alpha_t}{1 - 2\nu} \frac{1}{\gamma T_0} \int \int T \left[ \lambda_0 \nabla^2 T - c_T \frac{\partial T}{\partial t} \right] dV. \quad (31)
\]
The first term on the left-hand side of (31) can be transformed by using the divergence theorem as follows

\[
\iint_V T \nabla^2 T \, dV = \iint_V \int \frac{\partial}{\partial x_i} \left[ T \frac{\partial T}{\partial x_i} \right] \, dV - \iint_V \int \left( \frac{\partial T}{\partial x_i} \right) \left( \frac{\partial T}{\partial x_i} \right) \, dV
\]

\[
= \int_S T \frac{\partial T}{\partial x_i} n_i \, dS - \iint_V \int \left( \frac{\partial T}{\partial x_i} \right) \left( \frac{\partial T}{\partial x_i} \right) \, dV
\]

\[
= \frac{1}{\lambda_0} \int_V T \hat{q}_i n_i \, dS - \iint_V \int \left( \frac{\partial T}{\partial x_i} \right) \left( \frac{\partial T}{\partial x_i} \right) \, dV. \tag{32}
\]

From the boundary condition (26) we conclude that the first term on the right-hand side is equal to zero, so that (31) becomes

\[
\frac{\partial}{\partial t} \left\{ u \iint_V \left[ E_{ij} E_{ij} + \frac{\nu}{1-2\nu} \nabla^2 \right] \, dV + \frac{1}{2} \iint_V \int \rho_0 \left( \frac{\partial u_i}{\partial t} \right) \left( \frac{\partial u_i}{\partial t} \right) \, dV \right\}
\]

\[
+ \frac{E\alpha_i}{1-2\nu} \gamma \int \frac{\partial}{\partial t} \int T^2 \, dV = -\frac{E\alpha_t}{1-2\nu} \gamma \lambda_0 \int \frac{\partial}{\partial x_i} \int \frac{\partial T}{\partial x_i} \left( \frac{\partial T}{\partial x_i} \right) \, dV. \tag{33}
\]

The local form of the second law of thermodynamics implies that the heat flux and the temperature gradient must satisfy

\[
\frac{\hat{q}_i}{\partial T} \geq 0. \tag{34}
\]

In our case \( \hat{q}_i = \lambda_0 \partial T / \partial x_i \), so that (34) implies \( \lambda_0 > 0 \) and we assume that this condition holds. Therefore (33) leads to

\[
\frac{\partial}{\partial t} \left\{ \mu \iint_V \left[ E_{ij} E_{ij} + \frac{\nu}{1-2\nu} \nabla^2 \right] \, dV + \frac{1}{2} \iint_V \int \rho_0 \left( \frac{\partial u_i}{\partial t} \right) \left( \frac{\partial u_i}{\partial t} \right) \, dV \right\}
\]

\[
+ \frac{E\alpha_t}{1-2\nu} \gamma \int \frac{\partial}{\partial t} \int T^2 \, dV \right\} \leq 0. \tag{35}
\]

The inequality (35) is satisfied at \( t = t_0 \) with the equality sign. The expression in parentheses is a positive definite quadratic function \( \hat{S} \) of components of the velocity, strain tensor, and temperature. Thus we write (35) as

\[
\frac{d\hat{S}}{dt} \leq 0; \quad \hat{S}(t = t_0) = 0. \tag{36}
\]

---

6In general \( \hat{q}_i = k_{ij} \partial T / \partial x_j \) where \( k_{ij} \) is the thermal conductivity tensor.
From (36) it follows that $\bar{S}$ is a decreasing function. Using (36) we conclude that $\bar{S} \leq 0$. Since $\bar{S}$ is positive definite we have $\bar{S} \geq 0$. Thus

$$\bar{S} = \mu \int \iiint_V \left[ E_{ij} E_{ij} + \frac{v}{1 - 2\nu} \varphi^2 \right] dV + \frac{1}{2} \int \iiint_V \rho_0 \left( \frac{\partial u_i}{\partial t} \right) \left( \frac{\partial u_i}{\partial t} \right) dV$$

$$+ \frac{E\alpha_t}{1 - 2\nu} \frac{c_e}{\gamma T_0} \frac{1}{2} \int \iint_V T^2 dV = 0,$$  \hspace{1cm} (37)

for all $t \geq t_0$. From (37) we conclude that the differences of solutions are equal to zero; that is,

$$\frac{\partial u_i}{\partial t} = 0; \quad E_{ij} = 0; \quad T = 0,$$  \hspace{1cm} (38)

for all $t \geq t_0$. Thus, the solution of the boundary value problem (21),(22), (23) is unique up to the rigid body motion. If the displacement is specified on $S$ so that the rigid body motions are excluded, then the solution is unique.

### 4.5 Assumptions about solution of equilibrium equations

There is no general method to solve the equations of equilibrium for an elastic body. In many practical situations special assumptions about the form of the solution ("ansatz") are made. Usually this form contains one or more auxiliary functions that satisfy part of the complete system of equilibrium equations identically. The auxiliary functions are specified within their assumed class by forcing them to satisfy the rest of the conditions of the system of equilibrium equations. Depending on the equations that are used (Lamé or Beltrami-Michell) the assumptions are either about the displacement vector or about stress components. We present a few such procedures.

**A. Assumptions about displacement**

For the solution of Lamé equations (4.1-8) we introduce the assumption about the displacement vector. Here are concrete procedures.

**A.1 Galerkin’s vector**

It is known that any vector field (in our case $u$) can be represented as a sum of two vector fields, say $v$ and $w$, such that $\text{div} \ v = 0$ and $\text{curl} \ w = 0$. 
A necessary condition for the existence of such \( v \) and \( w \) is the existence of a scalar function \( \varphi \) and a vector field \( \Psi \) such that \( v = \text{curl} \, \Psi = \nabla \times \Psi \) and \( w = \text{grad} \, \varphi \). Using this, we assume that the displacement field is given as

\[
2\mu u = \text{grad} \, \varphi + \nabla \times \Psi. \tag{1}
\]

Suppose further that the term \( \text{curl} \, \Psi = \nabla \times \Psi \) could be expressed in terms of a new vector function \( \Phi = F_i e_i \) that satisfies

\[
(\nabla \times \Psi)_i = c \left[ \frac{\partial}{\partial x_j} \left( \frac{\partial F_i}{\partial x_j} \right) - \frac{\partial}{\partial x_j} \left( \frac{\partial F_j}{\partial x_i} \right) \right], \tag{2}
\]

where \( c \) is a constant determined later. Then (1) becomes

\[
2\mu u_i = \frac{\partial \varphi}{\partial x_i} + c \left[ \frac{\partial^2 F_i}{\partial x_j \partial x_j} - \frac{\partial^2 F_j}{\partial x_i \partial x_j} \right]. \tag{3}
\]

Since

\[
\frac{\partial}{\partial x_j} \left( \frac{\partial F_j}{\partial x_i} \right) = \frac{\partial F_1}{\partial x_1 \partial x_i} + \frac{\partial F_2}{\partial x_2 \partial x_i} + \frac{\partial F_3}{\partial x_3 \partial x_i} = \frac{\partial}{\partial x_i} \text{div} \, \Phi, \tag{4}
\]

the last term on the right-hand side of (3) represents a gradient of a scalar \( \frac{\partial F_i}{\partial x_i} = \text{div} \, \Phi \). We take scalar \( dvi \, \Phi \) and combine it with the scalar \( \varphi \) and then redefine \( \Phi \), so that (1) becomes

\[
2\mu u_i = cF_{i,ij} - F_{j,ij}. \tag{5}
\]

By substituting (5) into (4.1-9) with \( f_{i,0} \), we obtain

\[
(1 - 2\nu)[cF_{i,jj} - F_{j,ij}] + [cF_{k,jj} - F_{j,kj}], \tag{6}
\]

Recall that in (6) we used \( (\cdot),i = \partial(\cdot)/\partial x_i \). From (6) it follows that

\[
cF_{i,jj} + \left( \frac{c - 1}{1 - 2\nu} - 1 \right) F_{j,kj} = 0. \tag{7}
\]

If we choose \( c \) so that

\[
\frac{c - 1}{1 - 2\nu} - 1 = 0, \tag{8}
\]

or \( c = 2(1 - \nu) \) then the displacement vector given by (5), becomes (with \( F_i = H_i \))

\[
2\mu u_i = 2(1 - \nu)H_{i,ij} - H_{j,ij}. \tag{9}
\]

The vector \( H = H_i e_i \) is called Galerkin's vector.
The components of $\mathbf{H}$ satisfy the biharmonic equation (this follows from (7) with $c$ chosen according to (8))

$$
\nabla^4 H_i = H_{i,jjkk} = \frac{\partial^4 H_i}{\partial x_1^4} + \frac{\partial^4 H_i}{\partial x_2^4} + \frac{\partial^4 H_i}{\partial x_3^4} + 2 \frac{\partial^4 H_i}{\partial x_1^2 \partial x_2^2} + 2 \frac{\partial^4 H_i}{\partial x_1^2 \partial x_3^2} + 2 \frac{\partial^4 H_i}{\partial x_2^2 \partial x_3^2} = 0. \tag{10}
$$

If we find Galerkin's vector from (10) the displacement vector is determined by (9) and we can easily find components of the stress tensor.

A.2. Lamé potential. Love function

In the special case when the components of Galerkin's vector are not only biharmonic $\nabla^4 H_i = 0$ but also harmonic functions

$$
\nabla^2 H_i = 0, \tag{11}
$$

the expression (9) becomes

$$
2\mu u_i = -H_{j,ji}. \tag{12}
$$

We introduce next the displacement potential $\phi$ as

$$
\phi = H_{j,j} = \text{div } \mathbf{H} = \frac{\partial H_1}{\partial x_1} + \frac{\partial H_2}{\partial x_2} + \frac{\partial H_3}{\partial x_3}, \tag{13}
$$

so that (12) becomes

$$
2\mu u_i = -\phi, \tag{14}
$$

The scalar $\phi$ is called the Lamé displacement potential.

Suppose that Galerkin's vector has only one component, say $H_3 = Z$. Then

$$
\nabla^4 Z = 0. \tag{15}
$$

In this case $Z$ is called the Love displacement function. Note that (9) leads to

$$
2\mu u_1 = -\frac{\partial^2 Z}{\partial x_1 \partial x_3}; \quad 2\mu u_2 = -\frac{\partial^2 Z}{\partial x_2 \partial x_3};
$$

$$
2\mu u_3 = 2(1 - \nu) \left[ \frac{\partial^2 Z}{\partial x_1^2} + \frac{\partial^2 Z}{\partial x_2^2} + \frac{\partial^2 Z}{\partial x_3^2} \right] - \frac{\partial^2 Z}{\partial x_3^2}. \tag{16}
$$

A.3. Papkovich-Neuber functions

In some problems the displacement vector is assumed in the form of a combination of harmonic functions

$$
u_i = A(\phi_0 + x_j \phi_j)_i + B\phi_i, \tag{17}$$
4.5 Assumptions about solution of equilibrium equations

where $A$ and $B$ are constants and

$$
\nabla^2 \phi_0 = 0; \quad \nabla^2 \phi_i = 0; \quad i = 1, 2, 3. \tag{18}
$$

As is seen from (17) the displacement vector is expressed in terms of four harmonic functions.

The assumption (17), (18) was used by Papkovich and Neuber, independently. Mindlin has shown that there is a connection between the Galerkin vector and Papkovich-Neuber functions. Namely, suppose that the Laplacian ($\nabla^2 H_i$) and divergence ($\partial H_i / \partial x_i$) of the components of Galerkin’s vector are given as

$$
H_{i,jj} = \nabla^2 H_i = 2\tilde{\phi}_i; \quad H_{i,i} = \text{div} \ H_i = \psi, \tag{19}
$$

where $\tilde{\phi}_i$ and $\psi$ are some functions to be determined. Then, from (10) we obtain

$$
\nabla^4 H_i = H_{i,jjkk} = 2\tilde{\phi}_{i,kk} = 2\nabla^2 \tilde{\phi}_i = 0, \tag{20}
$$

that is, $\tilde{\phi}_i$ are harmonic functions. Also, from (19) it follows that

$$
H_{i,jj} = 2\tilde{\phi}_{i,i} = \psi_{,jj} = \nabla^2 \psi. \tag{21}
$$

By applying Laplace operator $\nabla^2 (\cdot)$ on both sides of (21) we conclude that

$$
\nabla^4 \psi = \psi_{,ijjj} = 0, \tag{22}
$$

that is, $\psi$ is a biharmonic function. From (21) we get

$$
\nabla^2 \psi = \psi_{,ii} = 2\tilde{\phi}_{i,i}, \tag{23}
$$

so that

$$
\psi = x_i \tilde{\phi}_i + \phi_0, \tag{24}
$$

where $\phi_0$ is an arbitrary harmonic function. Thus relation (24) that follows from the special assumption (19) about Galerkin’s vector, leads to the displacement potential (see (13), (14), and (19)2) of the form (17). By using (19) and (24) in (9) we obtain

$$
2\mu u_i = 4(1 - \nu)\tilde{\phi}_i - (x_m \tilde{\phi}_m + \phi_0),_i = (3 - 4\nu)\tilde{\phi}_i - x_j \tilde{\phi}_{j,i} - \phi_{0,i}, \tag{25}
$$

where, as stated, $\phi_0$ and $\tilde{\phi}_i$, $i = 1, 2, 3$ are harmonic functions.

In certain cases ($\nu \neq 1/4$, restrictions on the domain) it is possible to use only three harmonic functions in the representation (25). For such analysis see Sokolnikoff (1956).

---

B. Assumptions about stresses

For the solution of equilibrium equations expressed in terms of stresses we have a system consisting of (1.3-10) and Beltrami-Michell equations (3.7-21) that, in the case of zero body forces, reads

\[ \sigma_{ij,j} = 0; \quad \nabla^2 \sigma_{ij} + \frac{1}{1 + \nu} \Theta_{,ij} = 0, \]  

(26)

usually the stress functions method is used. We present a few possibilities for the choice of stress functions.

B.1. Maxwell’s stress functions

Let \( U_1, U_2, \) and \( U_3 \) be functions such that the stress components are given as

\[ \sigma_{11} = U_{2,33} + U_{3,22}; \]
\[ \sigma_{22} = U_{3,11} + U_{1,33}; \]
\[ \sigma_{33} = U_{1,22} + U_{2,11}; \]
\[ \sigma_{12} = -U_{3,12}; \quad \sigma_{23} = -U_{1,23}; \quad \sigma_{13} = -U_{2,13}. \]  

(27)

With (27) the equilibrium equations (26) are satisfied identically. By using (27) in (26) we obtain

\[ [(1 + \nu) \nabla^2 U_2 - \Theta],_{33} + [(1 + \nu) \nabla^2 U_3 - \Theta],_{22} = 0; \]
\[ [(1 + \nu) \nabla^2 U_1 - \Theta],_{22} + [(1 + \nu) \nabla^2 U_2 - \Theta],_{11} = 0; \]
\[ [(1 + \nu) \nabla^2 U_3 - \Theta],_{11} + [(1 + \nu) \nabla^2 U_1 - \Theta],_{33} = 0; \]
\[ -[(1 + \nu) \nabla^2 U_3 - \Theta],_{12} = 0; \]
\[ -[(1 + \nu) \nabla^2 U_2 - \Theta],_{13} = 0; \]
\[ -[(1 + \nu) \nabla^2 U_1 - \Theta],_{23} = 0. \]  

(28)

In writing (28) we used (4.1-17). From (28) we can obtain three independent relations that the functions \( U_i \) must satisfy. The functions \( U_i \) are called the Maxwell stress functions.

B.2. Finzi stress functions

Maxwell stress functions are often hard to obtain. Finzi proposed the stress functions in the form of the stress function tensor. Namely, suppose
that functions $U_{ij}$, $i, j = 1, 2, 3$ are given such that the following relations hold (see Finzi (1934))

\[
\begin{align*}
\sigma_{11} &= U_{22,33} + U_{33,22} - 2U_{23,23}; \\
\sigma_{22} &= U_{11,33} + U_{33,11} - 2U_{31,31}; \\
\sigma_{33} &= U_{11,22} + U_{22,11} - 2U_{12,12}; \\
\sigma_{12} &= [U_{23,1} + U_{31,2} - U_{12,3},3 - U_{33,12}]; \\
\sigma_{13} &= [U_{23,1} + U_{31,2} + U_{12,3},2 - U_{22,13}]; \\
\sigma_{23} &= [-U_{23,1} + U_{31,2} + U_{12,3},1 - U_{11,23}].
\end{align*}
\]

with

\[
U_{ij} = U_{ji}.
\]

The functions $U_{ij}$ are used as components of a second-order tensor $U$. They are called the Finzi stress functions. Relation (29) could be written as

\[
\sigma_{ij} = \epsilon_{imr}\epsilon_{jns}U_{rs,mn},
\]

where $\epsilon_{ijk}$ is the Levi-Civita permutation symbol (for a definition see Problem 12 of Chapter 2). It could be easily shown that (29) satisfies equilibrium equations (26) identically so that the only further condition that the six functions $U_{ij}$ have to satisfy are compatibility equations (26).

In plane problems of elasticity theory the Airy stress function is often used. It could be obtained from Maxwell stress functions if we set $U_1 = U_2 = 0$ and $U_3 \neq 0$.

### 4.6 Methods of solution

In previous sections we stated systems of elasticity theory equations. We distinguish three methods of solution of those equations as follows.

The direct method of solution is the method in which we seek the stress field, strain field, and the displacement field produced by given body and surface forces. Generally speaking, the direct method of solution leads to great mathematical difficulties.

In the inverse method we assign either the displacement field or the stress field in the body and determine all other quantities, including body and surface forces, that correspond to the assigned field. In the inverse method there are no great mathematical difficulties but it is often hard to obtain solutions that have practical interest.

In the semi-inverse method (or Saint-Venant method) one partially specifies stresses and (or) displacements and then uses the equations of the theory of elasticity to determine equations that must be satisfied by the remaining (not specified) stresses and (or) displacements. The semi-inverse
method has been successfully used in many important engineering problems. As a matter of fact, using this method Saint-Venant obtained the solutions, now considered to be classic, of bending and torsion problems for a prismatic rod.

There are numerous mathematical procedures that are used for solving systems of equations in each of the three methods specified. All procedures can be classified in these groups:

i) procedures in which exact analytical solution is obtained;
ii) procedures in which approximate analytical solution is obtained; and
iii) procedures in which numerical solution is obtained.

Of course in each concrete situation our goal is to obtain solution by procedures of group i). We briefly state some of the mathematical procedures. The detailed exposition of mathematical techniques can be found in specialized books, for example, Parton and Perlin (1984).

1. The complex variable method

For solution of plane problems of elasticity theory the complex variable method is used successfully. It was developed by G. V. Kolosov and N. I. Muskhelishvili (see Muskhelishvili (1966)). It is based on the special form of solution of a biharmonic equation. Namely, in two dimensions any solution of a biharmonic equation can be represented in terms of two analytic functions of a complex variable. This solution is then used to express stresses within the body. We present the complex variable method in Section 6.4.

2. The integral transform method

In this method a number of independent variables are reduced by applying integral transforms to the relevant system of equations. The transformed system is then solved (in most cases analytically). Finally the original solution is obtained by applying the inverse transform. To be specific, let \( \mathcal{F}(\lambda) \) be the transform of a function \( f(x) \). Then

\[
\mathcal{F}(\lambda) = \int_a^b f(x)K(x, \lambda)dx,
\]

where \( K(x, \lambda) \) is the kernel of the transformation and the limits of integration \( a \) and \( b \) could be finite or infinite. The most often used transformations are

\[
\text{Laplace} \quad K(x, \lambda) = e^{-\lambda x},
\]

\[
\text{Fourier} \quad K(x, \lambda) = e^{i\lambda x}, \quad i = \sqrt{-1};
\]
4.6 Methods of solution

\[ Mellin \quad : \quad K(x, \lambda) = x^{\lambda-1}; \]

\[ Bessel \quad : \quad K(x, \lambda) = J_n(\lambda x)x, \]  

where \( J_n \) is the Bessel function of the first order of the index \( n \).

3. The power and Fourier series methods

The solution of elasticity equations is often assumed in the form of a
power or Fourier series. For example, in the plane problems, the stress
function may be assumed in the form of a finite power series

\[ \Phi = \sum_{i=1}^{n} a_i x_1^i x_2^{n-i}. \]  

The solution of the differential equation for the stress function can be as­
sumed in the form of an infinite series

\[ \Phi = \sum_{n=1}^{\infty} f_n(x_1) \varphi_n(x_2). \]  

In (4) the functions \( f_n \) and \( \varphi_n \) are specially chosen. For example, \( \varphi_n \) is
taken as

\[ \varphi_n(x_2) = C_n \sin(\lambda_n x_2) + D_n \cos(\lambda_n x_2), \]  

where \( C_n, D_n, \) and \( \lambda_n \) are constants.

As an example of the mathematical method used to obtain analytical
approximation of the solution (group ii) we mention

4. The Ritz method

The Ritz method belongs to the class of direct methods of variational
calculus (see Dacorogna (1989) and Vujanovic and Jones (1989)). The so­
lation of the relevant equation in this method is assumed in a certain class
of functions, say \( U \). Then it is proved that the exact solution of the elastic­
ity equations for a given problem gives a minimum to a functional (usually
the total energy of the body). A specific element from class \( U \) is then cho­
sen so that it gives the smallest possible (within the class of functions \( U \))
value of the functional. By approximating the value of the functional with
the function from class \( U \) we hope to obtain the approximation of the exact
minimizing element of the functional that represents the exact solution of
the elasticity equations. The approximate solutions obtained by the Ritz
method often have an advantage over the other analytical approximate so­
lutions since for the Ritz solution we can, sometimes, derive an estimate of
the error of the approximate solution.
The Ritz method has an important application in the finite element methods and its numerous variants. The error estimates for Ritz method could be obtained from the corresponding variational principles.

From the methods of group iii) we mention

5. The finite element and boundary elements methods

Both methods belong to numerical methods. Their main characteristic is that the values of the dependent variables within the body are substituted by their values at discrete points within the body and (or) on its boundary. The values of the functions at discrete points are then obtained by a numerical procedure where the specific functional is minimized (Ritz type of procedure) or set to zero (weighted residuals method). In any case as a result of calculation one obtains the numerical values of the dependent variables at a finite number of points. Both methods are treated in many specialized books (see Zinkiewicz and Cheung (1967) and Brebbia and Walker (1980)). Also there exist numerous software packages that use the finite element and the boundary element methods for the solution of field problems (fluid mechanics, elasticity, etc.).

6. The finite difference method

This is a classical numerical method for solving ordinary and partial differential equations. Numerical values of dependent variables (stresses, strains, displacements) are obtained. The problem of establishing the convergence of the numerical procedure and the behavior of the solution at singular points are the main points where special care must be taken in using this method.

4.7 Saint-Venant principle

From the analysis presented so far it must be clear that the problem of solving the differential equations of elasticity theory represents a serious mathematical task. It is often very difficult to obtain a solution that satisfies the differential equations and the boundary and initial conditions. Often the solution is obtained more easily if we change the boundary conditions. The change over the part of the boundary is such that the resultant force and resultant couple of the "new" boundary conditions are the same as the resultant force and resultant couple of the "old" (initial, original) boundary conditions. The important question is then how much do the stress, strain, and the displacement within the body differ as a consequence of the change of the boundary conditions from "old" to "new"? The answer to this question is given by the famous Saint-Venant principle: if some distribution of surface forces on a portion of the surface of the body
is replaced by a different distribution, having the same resultant force and resultant couple, then the effect of both surface forces at the points of the body sufficiently removed from the region of application of the forces is the same. The physical meaning of the Saint-Venant principle is explained for the rod shown in Fig. 2.

Suppose that a rod, shown in the figure, is built in a rigid wall. Let $S_1$ be the part of the surface of the rod that is in contact with the wall. The stress vector distribution over $S_1$ is not known in advance and, as a matter of fact, is very complicated. However, from the global equilibrium conditions (see Section 1.3) we conclude that the action of the wall is (statically) equivalent to a concentrated force $F$ and a concentrated couple $M$. The Saint-Venant principle states that the deformation of the rod and the stress field in it at the points that are "sufficiently removed" from $S_1$ are the same, no matter what boundary conditions in the elasticity equations we use as long as the resultant force and resultant couple are equal to $F$ and $M$, respectively. Thus, in application, the Saint-Venant principle is appealed to whenever we want to neglect the local and end effects.

The importance and mathematical implications of the Saint-Venant principle are best understood if we, as Saint-Venant originally did, consider the deformation in an elastic rod. Since the difference between the resultant force and resultant couple of "old" and "new" boundary conditions is zero, we may consider the stress and deformation field in a prismatic elastic rod of length $L$ that is produced by the external load having the resultant force and resultant couple equal to zero. This stress and displacement field

---

will give us a quantitative representation of the error that is introduced by using the Saint-Venant principle. We present the final results of the work of Toupin (1965) and Parton and Perlin (1984) where the references to other important results are given. These results may be stated as:

i) if a system of forces is applied at the points of a surface that is inside a sphere radius $\epsilon$ and if this system of forces has the resultant force equal to zero, it causes stresses and strains of the order $\epsilon$ at all internal points of the sphere;

ii) if in addition, the sum of moments of the forces is equal to zero the order of stresses and strains will also be $\epsilon$;

iii) if the sum of forces and sum of moments are equal to zero and some additional conditions are satisfied (the so-called polymoments are equal to zero) then the stresses and strains are of the order $\epsilon^2$.

For recent results on the Saint-Venant problem we refer the reader to review articles by Horgan and Knowels (1983) and Horgan (1989).

**Problems**

1. By using Maxwell stress functions (4.5-27) show that the equilibrium equations (4.5-26) are satisfied.

2. Show that each of the following functions satisfies the biharmonic equation (4.5-15).
   
   a) $(C_1e^{\alpha x_1} + C_2e^{-\alpha x_1} + C_3x_1e^{\alpha x_1} + C_4x_1e^{-\alpha x_1})\sin(\alpha x_2),
      
      \quad C_i = \text{const.}, \quad \alpha = \text{const.},$
   
   b) $x_1V,$
   
   c) $x_2V,$
   
   d) $(x_1^2 + x_2^2)V,$

   where $V$ is a harmonic function; that is, $\nabla^2V = 0.$

3. By using Hooke's law show that the displacement vector could be expressed in terms of the Maxwell stress functions as

   \[ u_1 = \frac{1+\nu}{E}(U_1 - U_2 - U_3),_1; \quad u_2 = \frac{1+\nu}{E}(U_2 - U_1 - U_3),_2; \quad u_3 = \frac{1+\nu}{E}(U_3 - U_1 - U_2),_3. \]
4. Suppose that the body forces are described by a trigonometric function of time
\[ f = f^{(1)}(x_i) \cos \omega t + f^{(2)}(x_i) \sin \omega t, \quad (a) \]
where \( \omega \) is a constant and \( f^{(1)} \) and \( f^{(2)} \) are given functions. The dynamic problem with forces in this form is called the problem of periodic oscillations with frequency \( \omega \). Assume solutions for displacement as
\[ u(x_j, t) = u^{(1)}(t) \cos \omega t + u^{(2)}(t) \sin \omega t. \quad (b) \]
If the complex amplitude of periodic oscillations \( u_c = (u_1, u_2, u_3) \) is defined as
\[ u_c = u^{(1)} + i u^{(2)}, \quad i = \sqrt{-1}, \quad (c) \]
and the complex body force as
\[ f_c = f^{(1)} + i f^{(2)}, \quad (d) \]
then by using (4.3-1) show that the complex amplitude vector satisfies
\[ (\lambda + \mu) \nabla \text{div} u_c + \mu \nabla^2 u_c + (f_c + \rho_0 \omega^2 u_c) = 0. \quad (e) \]

5. Show that for the case when the body forces are not equal to zero equation (4.5-10) becomes
\[ H_{i,j,k,k} = -\frac{f_i}{1 - \nu}. \]
It is interesting that not only when the body forces are absent but also in the case when they are not equal to zero, the individual components of Galerkin's vector satisfy equations that are not coupled.

6. Consider the static case in (4.3-1). Define the operator of elasticity as
\[ A(u) = (\lambda + 2\mu) \nabla \text{div} u - \mu (\text{curl} \text{curl} u). \quad (a) \]
Show that the static case of (4.3-1) corresponds to
\[ A(u) + f = 0. \quad (b) \]
The operator of elasticity is a symmetric differential operator. It is positive for the case of second and third fundamental boundary value problems (see Parton and Perlin (1984, vol. II p. 293)). Note also that for \( \mu = 1, \lambda = -1 \) the operator \( A \) becomes the Laplace operator.
Chapter 5

Solutions for Some Problems of Elasticity Theory

5.1 Introduction

In this chapter we present solutions of a few concrete problems of the theory of elasticity using the semi-inverse method. All problems have practical application in engineering. We illustrate the type of problems that can be solved by the "classical" methods of the previous chapter. Also, we write some solutions in a form that allows comparison with the solutions obtained by elementary strength of materials and rod theories. Since in engineering applications the rod is, probably, one of the most often used elements, we treat, in detail, important problems of extension, torsion, and bending formulated for an elastic rod. These problems (determination of the stress strain and deformation state of a prismatic rod loaded by a system of forces and couples applied at the ends of the rod) are called the Saint-Venant problems.

The next important type of problem is the class of problems for cylinders and spheres. Namely, we analyze the stress and deformation fields for elastic and thermoelastic cylinders and spheres subjected to surface and body forces.

Also, we present the solution to Boussinesq problems (elementary solutions of the first and second kind). In solving the problem of concentrated force acting on an elastic body we are faced with the notion of generalized functions or distributions. The best known generalized function is the Dirac delta function. Detailed analyzes of generalized functions are given in specialized books (see, for example, Vladimirov (1979) and Lighthill (1980)). We give here the physical motivation for the introduction of such functions. The generalized functions reflect the fact that we cannot measure a specific physical quantity, in our case force, at a given point. Instead we measure mean values of force over a small neighborhood of the point and then proclaim the limit of those mean values as the value of the quantity at the point.

Suppose we want to determine the intensity of a concentrated force $F$
acting at the origin of a rectangular coordinate system positioned at the outer surface $S$ of the body with axis $x_3$ oriented along the normal to $S$. Suppose further that the intensity of the concentrated force is $F = 1$.

To measure $F$ we uniformly distribute the force over a circular area, of radius $\epsilon$ in the $x_1 - x_2$ coordinate plane centered at the point $x_1 = x_2 = x_3 = 0$. Then we obtain the *mean* value of the force over the entire surface $S$ of the body given by the function

$$F_\epsilon(x_1, x_2) = \begin{cases} \frac{1}{\pi \epsilon^2} & \text{if } (x_1^2 + x_2^2) < \epsilon \\ 0 & \text{if } (x_1^2 + x_2^2) > \epsilon \end{cases}.$$  \hfill (1)

Note that

$$\int_S F_\epsilon dA = 1.$$  \hfill (2)

We are interested in the limit value for $\epsilon \to 0$. We denote this point-wise limit by $\delta(x_1, x_2)$. From (1) it follows that

$$\delta(x_1, x_2) = \begin{cases} +\infty & \text{if } (x_1^2 + x_2^2) = 0 \\ 0 & \text{if } (x_1^2 + x_2^2) \neq 0 \end{cases}.$$  \hfill (3)

Also since the intensity of the force is equal to one, we require that

$$\int_S \delta(x_1, x_2) dA = 1.$$  \hfill (4)

However the function defined by (3) has $\int_S \delta dS = 0$. This means that (3) does not reproduce the value of the force. To avoid this problem we change the type of limit taken in obtaining (3). Thus, instead of taking a pointwise limit of $F_\epsilon$ we use notion of the *weak limit*. Namely, for any continuous function $\phi(x_1, x_2)$ the weak limit of $F_\epsilon$ is

$$\lim_{\epsilon \to +0} \int_S F_\epsilon(x_1, x_2) \phi(x_1, x_2) dS = \int_S \delta(x_1, x_2) \phi(x_1, x_2) dS = \phi(0, 0).$$  \hfill (5)

Thus, the weak limit of $F_\epsilon$ is a functional assigning the value $\phi(0, 0)$ to a function $\phi(x_1, x_2)$. We say that the Dirac $\delta$ function is a weak limit of the sequence $F_\epsilon$; that is,

$$\lim F_\epsilon \Rightarrow \delta.$$  \hfill (6)

Note that from (4) it follows that

$$\int_S \delta(x_1, x_2) dS = 1,$$  \hfill (7)
so that (4) holds when $\delta$ is a weak limit of $F$. In the case when there are several concentrated forces of intensities $F_i$ acting at the points $1, 2, \ldots, K$, with the coordinates $x_1^1, x_2^1, x_3^1, x_1^2, \ldots, x_3^K$, the force distribution function becomes

$$F = \sum_{n=1}^{K} F_n \delta((x_1 - x_1^n), (x_2 - x_2^n), (x_3 - x_3^n)).$$  \hfill (8)

Instead of (1) we could use different functions, called the basic functions (or “good functions”). For example, the cup function is defined as

$$\omega_\epsilon(x_1, x_2) = \begin{cases} C_\epsilon \exp\left(-\epsilon^2/(\epsilon^2 - |x|^2)\right), & \text{if } |x|^2 = (x_1^2 + x_2^2) \leq \epsilon \\ 0, & \text{if } (x_1^2 + x_2^2) > \epsilon \end{cases},$$ \hfill (9)

and could be used as the basic function if the constant $C_\epsilon$ is determined from the condition (2), or

$$\omega_\epsilon = \frac{1}{\epsilon \sqrt{2\pi}} \exp\left(-\frac{1}{2\epsilon^2} |x|^2\right),$$ \hfill (10)

where, again, $|x|^2 = (x_1^2 + x_2^2)$. The function (10) satisfies (2). Note that the functions (9),(10) are continuous functions. They each can be used as a model for the Dirac $\delta$ function. The representation of concentrated forces by Dirac function is especially useful in the case when functional transformations are used for the solution of concrete problems (see, for example, Section 6.3).

This chapter is organized so that in the first part we treat static problems and in the last part we present solutions of thermoelastic and dynamic problems.

### 5.2 Heavy rod

Consider a prismatic elastic rod of length $l$ fixed at the upper end $x_3 = l$.

The precise manner in which the rod is fixed at $x_3 = l$ is specified later. At the lower end we introduce a Cartesian coordinate system $\hat{x}_i$.

The axis $\hat{x}_3$ is so oriented that it coincides with the geometrical axis of the rod and is directed upward (see Fig. 1). The lateral surface of the rod is assumed to be stress free. The components of the body forces in the system $\hat{x}_i$ are

$$f_1 = f_2 = 0; \quad f_3 = -\rho_0 g,$$ \hfill (1)

where $\rho_0$ is the mass density and $g$ is the gravity acceleration. Suppose that the stress tensor in the rod has components

$$\sigma_{33} = \rho_0 g x_3; \quad \sigma_{11} = \sigma_{22} = \sigma_{12} = \sigma_{13} = \sigma_{23} = 0.$$ \hfill (2)
The stress tensor with components (2) satisfy the equilibrium equations
\((4.1-1)\) with \(\frac{\partial^2 u_i}{\partial t^2} = 0\). Also (2) satisfies the compatibility equations
\((4.1-20)\). Since the rod is prismatic unit normal \(\mathbf{n}\) to the lateral surface (see
Fig. 2)

\[
\mathbf{n} = \cos \varphi \mathbf{e}_1 + \sin \varphi \mathbf{e}_2, \quad \text{(3)}
\]

where \(\varphi\) is arbitrary. From (2) and (3) it follows that

\[
\mathbf{p}_n = 0. \quad \text{(4)}
\]

On the upper end of the rod \(x_3 = l\) we have

\[
\sigma_{33}(x_3 = l) = \rho_0 gl. \quad \text{(5)}
\]

The rod must be fixed at the upper end so that (5) holds. From Hooke’s
law (3.4-14) it follows that

\[
\begin{align*}
E_{11} &= \frac{1}{E}(-\nu \rho_0 g x_3); \quad & E_{22} &= \frac{1}{E}(-\nu \rho g x_3); \\
E_{33} &= \frac{1}{E}(\rho_0 g x_3); \quad & E_{12} &= E_{13} = E_{23} = 0. \quad \text{(6)}
\end{align*}
\]

Therefore, we have

\[
\begin{align*}
\frac{\partial u_1}{\partial x_1} &= \frac{\partial u_2}{\partial x_2} = -\frac{\nu \rho_0 g x_3}{E}; \quad & \frac{\partial u_3}{\partial x_3} &= \frac{\rho_0 g x_3}{E}; \\
\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2} + \frac{\partial u_3}{\partial x_3} &= \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} = 0. \quad \text{(7)}
\end{align*}
\]
From (7)_2 we obtain

\[ u_3 = \frac{\rho_0 g x_3^2}{2E} + w_0(x_1, x_2), \tag{8} \]

where \( w_0 \) is an arbitrary function of \( x_1 \) and \( x_2 \). From (7)_3 it follows that

\[ \frac{\partial u_1}{\partial x_3} = -\frac{\partial w_0}{\partial x_1}; \quad \frac{\partial u_2}{\partial x_3} = -\frac{\partial w_0}{\partial x_2}. \tag{9} \]

By integrating (9), we get

\[ u_1 = -x_3 \frac{\partial w_0}{\partial x_1} + u_0(x_1, x_2); \quad u_2 = -x_3 \frac{\partial w_0}{\partial x_2} + v_0(x_1, x_2), \tag{10} \]

where \( u_0 \) and \( v_0 \) are arbitrary functions of \( x_1 \) and \( x_2 \). If we now use (10) in (7)_1 we obtain

\[ \frac{\partial u_1}{\partial x_1} = -x_3 \frac{\partial^2 w_0}{\partial x_1^2} + \frac{\partial u_0}{\partial x_1} = -\frac{\nu \rho_0 g x_3}{E}; \]
\[ \frac{\partial u_2}{\partial x_2} = -x_3 \frac{\partial^2 w_0}{\partial x_2^2} + \frac{\partial v_0}{\partial x_2} = -\frac{\nu \rho_0 g x_3}{E}. \tag{11} \]

Since the right-hand sides of (11)\_1,2 are functions of \( x_3 \) it follows that

\[ \frac{\partial^2 w_0}{\partial x_1^2} = \frac{\nu \rho_0 g}{E}; \quad \frac{\partial^2 w_0}{\partial x_2^2} = \frac{\nu \rho_0 g}{E}; \quad \frac{\partial u_0}{\partial x_1} = 0; \quad \frac{\partial v_0}{\partial x_2} = 0. \tag{12} \]

By using (10) in (7)_3 we get

\[ \frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2} = -x_3 \frac{\partial^2 w_0}{\partial x_1 \partial x_2} + \frac{\partial u_0}{\partial x_1} - x_3 \frac{\partial^2 w_0}{\partial x_1 \partial x_2} + \frac{\partial u_0}{\partial x_2} = 0. \tag{13} \]

From (13) we conclude that

\[ \frac{\partial^2 w_0}{\partial x_1 \partial x_2} = 0; \quad \frac{\partial u_0}{\partial x_2} + \frac{\partial v_0}{\partial x_1} = 0. \tag{14} \]
Now (12) implies that \( u_0 = u_0(x_2) \) and \( v_0 = v_0(x_1) \). If we use this in (14) we obtain

\[
\frac{du_0}{dx_2} + \frac{dv_0}{dx_1} = 0, \tag{15}
\]

or

\[
\frac{du_0}{dx_2} = a = \text{const.}; \quad \frac{dv_0}{dx_1} = -a. \tag{16}
\]

Therefore, we have

\[
u_0 = as_2 + b_1; \quad v_0 = -ax_1 + b_2, \tag{17}
\]

where \( a, b_1, \) and \( b_2 \) are constants. From (12), and (14) we finally obtain

\[
w_0 = \frac{\nu \rho_0 g}{2E} (x_1^2 + x_2^2) + x_1 x_1 + x_2 x_2 + c_3, \tag{18}
\]

where \( c_1, c_2, \) and \( c_3 \) are constants. With (17) and (18) equations (8) and (10) become

\[
u_1 = -\frac{\nu \rho_0 g}{E} x_1 x_3 - c_1 x_3 + a x_2 + b_1;
\]

\[
u_2 = -\frac{\nu \rho_0 g}{E} x_2 x_3 - c_2 x_3 - a x_1 + b_2;
\]

\[
u_3 = \frac{\rho_0 g}{2E} (x_1^2 + \nu x_1^2 + \nu x_2^2) + c_1 x_1 + c_2 x_2 + c_3. \tag{19}
\]

The constants \( c_i, \ i = 1, 2, 3 \) we determine from the conditions at the upper end. If we require

\[
u_1 = \nu_2 = \nu_3 = 0; \quad \text{for} \quad x_1 = x_2 = 0; \quad x_3 = l, \tag{20}
\]

then the point \( M_0 \) (see Fig. 1) will not move. Furthermore, if we require that there is no possibility of rotation about the \( \bar{x}_1 \) axis we fix a surface element in the \( \bar{x}_1 - \bar{x}_3 \) plane at \( x_1 = x_2 = 0, x_3 = l \). This is equivalent to the requirement that \( \partial u_2 / \partial x_1 = 0 \) at \( x_1 = x_2 = 0; x_3 = l \). Similarly, if there is no possibility of rotation about the other two axes at the point \( x_1 = x_2 = 0, x_3 = l \) we require that \( \partial u_1 / \partial x_3 = 0 \) and \( \partial u_2 / \partial x_3 = 0 \). Thus (see (2.4-14)) the rotation vector is zero at the point \( M_0 \) if

\[
\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} = \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1} = \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} = 0, \tag{21}
\]

at \( x_1 = x_2 = 0, x_3 = l \). By using (19) in (20),(21) we obtain

\[
u_1 = -\frac{\nu \rho_0 g}{E} x_1 x_3; \quad u_2 = -\frac{\nu \rho_0 g}{E} x_2 x_3; \quad u_3 = \frac{\rho_0 g}{2E} (x_3^2 + \nu x_1^2 + \nu x_2^2 - l^2). \tag{22}
\]

From the solution (22) it is clear that the cross-sections of the rod that are plane in the undeformed state do not remain plane in the deformed state.
For example, a cross-section \(x_3 = c\) = const. in the undeformed state becomes a surface defined by

\[
x_3' = c + u_3 = c + \frac{\rho_0 g(c^2 - l^2)}{2E} + \frac{\nu \rho_0 g}{2E} (x_1^2 + x_2^2).
\]

(23)

Also it follows from (22) that all points on the lateral surface of the rod are displaced towards the axis of the rod and that the amount of this displacement is proportional to the distance from the lower end \(x_3\). The upper base \(x_3 = l\) is warped upward. This is a consequence of the boundary condition \(\sigma_{33} = \text{const.}\) at \(x_3 = l\).

5.3 Rotating rod

Consider an infinitely long circular rod having \(R\) as the outer radius. Suppose that the rod is rotating with constant angular velocity \(\omega\) about its axis. We consider the problem of relative equilibrium configuration of the rod, with respect to rotating (with angular velocity \(\omega\)) cylindrical coordinate system \((r, \theta, z)\). The axis \(z\) is oriented along the geometrical axis of the rod. The body force in this case is the inertial force so that it has only one component. In the equilibrium equations (1.5-6) we have

\[
f_r = \rho_0 \omega^2 r,
\]

(1)

where \(\rho_0\) is the mass density. Suppose further that

\[
\sigma_{r\theta} = \sigma_{zz} = \sigma_{zr} = \sigma_{z\theta} = 0,
\]

(2)

and that all dependent variables are functions of \(r\) only. Then, (1.5-6) leads to

\[
\frac{d \sigma_{rr}}{dr} + \frac{1}{r} \sigma_{rr} - \sigma_{r\theta} + \rho_0 \omega^2 r = 0.
\]

(3)

From the expression (3.6-1) we find

\[
\sigma_{rr} = \lambda \left(\frac{u}{r} + \frac{du}{dr}\right) + 2\mu \frac{du}{dr};
\]

\[
\sigma_{\theta\theta} = \lambda \left(\frac{u}{r} + \frac{du}{dr}\right) + 2\mu \frac{u}{r},
\]

(4)

where \(u_r = u\) and \(u_\theta = u_z = 0\). By using (4) in (3) we obtain

\[
\frac{d^2 u}{dr^2} + \frac{1}{r} \frac{du}{dr} - \frac{u}{r^2} + \rho_0 \frac{\omega^2 r}{\lambda + 2\mu} = 0.
\]

(5)

The solution of equation (5) reads

\[
u = C_1 r + \frac{c_2}{r} - \rho_0 \frac{\omega^2 r^3}{8(\lambda + 2\mu)}.
\]

(6)
where \( C_i, i = 1, 2 \) are constants. Since \( u(0) \) is finite, it follows that \( C_2 = 0 \) so that

\[
u = C_1 r - \rho_0 \frac{\omega^2 r^3}{8(\lambda + 2\mu)}.\] (7)

By using (7) in (4) we have

\[
\sigma_{rr} = 2C_1(\lambda + \mu) - 2(2\lambda + 3\mu) \frac{\rho_0 \omega^2 r^2}{8(\lambda + 2\mu)}.\] (8)

Since \( \sigma_{rr} (r = R) = 0 \) the constant \( C_1 \) becomes

\[
C_1 = \frac{2\lambda + 3\mu}{\lambda + \mu} \frac{\rho_0 \omega^2 R^2}{8(\lambda + 2\mu)}.\] (9)

With this value of \( C_1 \) the component of the displacement vector \( u \) is given as

\[
u = \frac{\rho_0 \omega^2 r}{8(\lambda + 2\mu)} \left( R^2 \frac{2\lambda + 3\mu}{\lambda + \mu} - r^2 \right).\] (10)

### 5.4 Spherical shell under inner and outer pressure

Consider a spherical shell of outer radius \( R_2 \) and inner radius \( R_1 < R_2 \). Let \( p_1 \) and \( p_2 \) be the inner and outer pressure acting on the shell, respectively. On the basis of symmetry, we assume that in the spherical coordinate system \( (\rho, \varphi, \theta) \),

\[
\sigma_{\theta\theta} = \sigma_{\varphi\varphi}(\rho); \quad \sigma_{\rho\rho} = \sigma_{\rho\rho}(\rho),\] (1)

and \( \sigma_{\rho\theta} = \sigma_{\rho\varphi} = \sigma_{\theta\varphi} = 0 \). Since the body forces are equal to zero, the equilibrium equations (1.5-9) reduce to a single equation

\[
\frac{d\sigma_{\rho\rho}}{d\rho} + \frac{1}{\rho} (2\sigma_{\rho\rho} - \sigma_{\theta\theta} - \sigma_{\varphi\varphi}) = 0.\] (2)

Further suppose that

\[
u_\rho = \nu_\rho(\rho); \quad \nu_\theta = \nu_\varphi = 0.\] (3)

Then, the strain tensor (the linear part in (2.6-11)) has the form

\[
E_{\rho\rho} = \frac{d\nu_{\rho}}{d\rho}; \quad E_{\varphi\varphi} = \frac{\nu_{\rho}}{\rho}; \quad E_{\theta\theta} = \frac{\nu_{\rho}}{\rho};
\]

\[
E_{\theta\varphi} = E_{\varphi\theta} = E_{\rho\theta} = 0.\] (4)

From Hooke's law in the form (3.6-2) and (4) we obtain

\[
\sigma_{\rho\rho} = \lambda \left( \frac{d\nu_{\rho}}{d\rho} + 2 \frac{\nu_{\rho}}{\rho} \right) + 2\mu \frac{d\nu_{\rho}}{d\rho};
\]

\[
\sigma_{\theta\theta} = \sigma_{\varphi\varphi} = \lambda \left( \frac{d\nu_{\rho}}{d\rho} + 2 \frac{\nu_{\rho}}{\rho} \right) + 2\mu \frac{\nu_{\rho}}{\rho}.\] (5)
By using (5) in (2) it follows that

\[
\frac{d^2 u_\rho}{d\rho^2} + 2 \frac{d}{d\rho} \left( \frac{u_\rho}{\rho} \right) = 0,
\]

or

\[
\frac{1}{\rho^2} \frac{d}{d\rho} (\rho^2 u_\rho) = \text{const.} = 3a.
\]

Solving (7) we obtain (with \( u = u_\rho \))

\[
u = a\rho + \frac{b}{\rho^2},
\]

where \( b \) is a new constant. By using (8) in (5) we get

\[
\sigma_{\rho\rho} = a(3\lambda + 2\mu) - 4\mu \frac{b}{\rho^3}; \\
\sigma_{\theta\theta} = a(3\lambda + 2\mu) + 2\mu \frac{b}{\rho^3}.
\]

(9)

The constants \( a \) and \( b \) are determined from the boundary conditions at the inner and outer surface of the shell. Thus, we have

\[
\sigma_{\rho\rho} = -p_1; \quad \text{for } \rho = R_1; \\
\sigma_{\rho\rho} = -p_2; \quad \text{for } \rho = R_2.
\]

(10)

By using (10) in (9) it follows that

\[
a = \frac{p_1 R_1^3 - p_2 R_2^3}{(3\lambda + 2\mu)(R_2^3 - R_1^3)}; \quad b = \frac{R_1^3 R_2^3 (p_1 - p_2)}{4\mu(R_2^3 - R_1^3)}.
\]

(11)

The displacement vector is completely determined with (8) and (11). The stress components (9) become

\[
\sigma_{\rho\rho} = \frac{1}{1 - \psi^3} \left[ \psi^3 \left( 1 - \frac{R_2^3}{\rho^3} \right) p_1 - \left( 1 - \frac{\psi^3 R_2^3}{\rho^3} \right) p_2 \right]; \\
\sigma_{\theta\theta} = \frac{1}{1 - \psi^3} \left[ \psi^3 \left( 1 + \frac{R_2^3}{2\rho^3} \right) p_1 - \left( 1 + \frac{\psi^3 R_2^3}{2\rho^3} \right) p_2 \right],
\]

(12)

where \( \psi = R_1/R_2 \). In the special case when the external pressure is zero, that is, \( p_2 = 0 \), equations (12) lead to

\[
\sigma_{\rho\rho} = \frac{1}{1 - \psi^3} \left[ \psi^3 \left( 1 - \frac{R_2^3}{\rho^3} \right) p_1 \right] = \frac{p_1 R_1^3}{R_2^3 - R_1^3} \left( 1 - \frac{R_2^3}{\rho^3} \right) < 0; \\
\sigma_{\theta\theta} = \frac{1}{1 - \psi^3} \left[ \psi^3 \left( 1 + \frac{R_2^3}{2\rho^3} \right) p_1 \right] = \frac{p_1 R_1^3}{R_2^3 - R_1^3} \left( 1 + \frac{R_2^3}{2\rho^3} \right) > 0.
\]

(13)
As another special case consider an infinite elastic body with spherical cavity of radius $R_1 = R$ in it. Suppose further that the body is loaded by uniform compression $p_2 = p$. By setting $R_2 \to \infty$, $p_1 = 0$, $p_2 = p$ in (11) we obtain
\[
a = \frac{-p}{(3\lambda + 2\mu)}; \quad b = \frac{-pR^3}{4\mu}, \tag{14}
\]
so that (9) leads to
\[
\sigma_{pp} = -p \left(1 - \frac{R^3}{\rho^3}\right); \quad \sigma_{\theta \theta} = -p \left(1 + \frac{R^3}{2\rho^3}\right). \tag{15}
\]
Note that on the boundary of the cavity $\rho = R$ we have
\[
\sigma_{pp} = 0; \quad \sigma_{\theta \theta} = -\frac{3}{2}p. \tag{16}
\]

### 5.5 Torsion of a prismatic rod with an arbitrary cross-section

Consider a prismatic rod of arbitrary (constant) cross-section and length $l$. The lateral surface of the rod is stress free and the bases of the rod are loaded by couples of intensity $M$ (or by a system of forces and couples having the resultant force equal to zero and resultant couple equal to $M$). There are no body forces acting. Let $\bar{x}_i$, $i = 1, 2, 3$ be a Cartesian coordinate system with the $\bar{x}_1$ axis oriented so that it is parallel to the geometrical axis of the rod. The origin of $\bar{x}_i$ is not positioned at any special point of the cross-section. It could, and often does, coincide with the centroid of the cross-section (see Fig. 3). We use the Saint-Venant method and make the following assumptions concerning the deformation of the rod.

i) The cross-section of the rod at $x_1 \neq 0$ rotates (relative to the cross-section at $x_1 = 0$) about the $\bar{x}_1$ axis for an angle that is proportional to the distance of the cross-section from the base $\bar{x}_1 = 0$.

ii) The component of the displacement vector along the $\bar{x}_1$ axis (i.e., $u_1$) is equal for all cross-sections (it is independent of $x_1$) and depends on $x_2$ and $x_3$ only.

From i) and ii) it follows that
\[
\varphi = \bar{\theta}x_1; \quad u_1 = u_1(x_2, x_3), \tag{1}
\]
where $\varphi$ is the rotation angle of the cross-section with respect to the cross-section at $x_1 = 0$ and $\bar{\theta}$ is a constant. From (1) we obtain the components of the displacement vector as (see Fig. 4)
\[
u_2 = \overline{OP}[\cos(\alpha + \varphi) - \cos \alpha]; \quad u_3 = \overline{OP}[\sin(\alpha + \varphi) - \sin \alpha]. \tag{2}
\]
If we assume that the angle $\varphi$ is small, then from (1)/2 and (2) it follows that

$$u_1 = u_1(x_2, x_3); \quad u_2 = -\bar{\theta}x_1 x_3; \quad u_3 = \bar{\theta}x_1 x_2. \quad (3)$$

By using (1) and (3) we obtain the components of the strain tensor as

$$E_{11} = E_{22} = E_{33} = E_{23} = 0;$$

$$E_{12} = \frac{1}{2} \left( -\bar{\theta}x_3 + \frac{\partial u_1}{\partial x_2} \right); \quad E_{13} = \frac{1}{2} \left( \bar{\theta}x_2 + \frac{\partial u_1}{\partial x_3} \right). \quad (4)$$

Equations (4) and Hooke's law (3.3-43) lead to

$$\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_{23} = 0;$$

$$\sigma_{12} = \mu \left( -\bar{\theta}x_3 + \frac{\partial u_1}{\partial x_2} \right); \quad \sigma_{13} = \mu \left( \bar{\theta}x_2 + \frac{\partial u_1}{\partial x_3} \right). \quad (5)$$

By using (5) the equilibrium equations become

$$\frac{\partial \sigma_{13}}{\partial x_1} = 0; \quad \frac{\partial \sigma_{12}}{\partial x_1} = 0; \quad \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} = 0. \quad (6)$$
If we write \( u_1 \) as
\[
    u_1 = \tilde{\phi}(x_2, x_3),
\]
where \( \Phi \) is a function to be specified, then by substituting (5) in (6) we obtain
\[
\nabla^2 \Phi(x_2, x_3) = 0,
\]
where
\[
\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x_2^2} + \frac{\partial^2 \Phi}{\partial x_3^2}.
\]
The boundary conditions corresponding to (9) we derive from the fact that the lateral surface of the rod is stress free. Thus
\[
    \mathbf{p}_n = (\sigma_{12}n_2 + \sigma_{13}n_3)e_1 = 0; \quad \text{on } C,
\]
where \( C \) denotes the boundary curve of the cross-section of the rod and
\[
    n_2 = \cos \angle(\mathbf{n}, \bar{x}_2); \quad n_3 = \cos \angle(\mathbf{n}, \bar{x}_3).
\]
From (5), (7), and (10) we obtain the following boundary condition for \( \Phi \) on \( C \),
\[
\left( \frac{\partial \Phi}{\partial x_2} - x_3 \right) n_2 + \left( \frac{\partial \Phi}{\partial x_3} + x_2 \right) n_3 = 0.
\]
The directional derivative of the function \( \Phi \) in the direction of the normal (called the normal derivative) to the curve \( C \) is defined as
\[
\frac{d\Phi}{dn} = \frac{\partial \Phi}{\partial x_2} n_2 + \frac{\partial \Phi}{\partial x_3} n_3,
\]
so that (12) becomes
\[
\frac{d\Phi}{dn} = x_3n_2 - x_2n_3; \quad \text{on } C.
\]
The function \( \Phi \) is called the torsion function and system (9),(14) represents the Neumann-type boundary value problem. Thus, the torsion function is harmonic inside the region bounded by \( C \) and has the normal derivative prescribed on the boundary. We integrate (14) along \( C \) to obtain
\[
\int_C \frac{\partial \Phi}{\partial n} dS = \int_C [x_3n_2 - x_2n_3] dS = \int_C [x_3 dx_3 + x_2 dx_2] = \int_C [x_2^2 + x_3^2] = 0.
\]
system (9), (14). Let $\Phi = \Phi^1 - \Phi^2$ be the difference of the solutions. The function $\Phi$ satisfies $\nabla^2 \Phi = 0$ inside the domain $A$ bounded by $C$ and $\partial \Phi / \partial n = 0$ on $C$. By using the Green theorem\(^2\) we have

$$\oint_C \frac{\partial \Phi}{\partial n} dS = \oint_C \left[ \frac{\partial \Phi}{\partial x_2} n_2 + \frac{\partial \Phi}{\partial x_3} n_3 \right] dS = \oint_C \left[ \frac{\partial \Phi}{\partial x_2} dx_3 - \frac{\partial \Phi}{\partial x_3} dx_2 \right]$$

$$= -\int_A \int [\nabla^2 \Phi] dA. \quad (16)$$

Since $\partial \Phi / \partial n = 0$ it follows that

$$\int_A \int [\nabla^2 \Phi] dA = 0, \quad (17)$$

so that $\Phi$ is a constant. Thus the torsion function is unique up to the constant.

The torsion moment could be determined as (see Fig. 5)

$$M = \int_A \int (\sigma_{13} x_2 - \sigma_{12} x_3) dA = \mu \bar{\theta} \int_A \int \left( x_2^2 + x_3^2 - \frac{\partial \Phi}{\partial x_2} x_3 + \frac{\partial \Phi}{\partial x_3} x_2 \right) dA. \quad (18)$$

\(^2\)If $P$ and $Q$ are functions that are continuous and have continuous partial derivatives in the domain $A$ bounded by a curve $C$, then $\int_A \int [\partial Q / \partial x_2 - \partial P / \partial x_3] dA = \oint_C [P dx_2 + Q dx_3] = \oint_C \mathbf{u} dS = \int_A \int \text{curl} \, \mathbf{u} dA$, where $\mathbf{u} = Pe_2 + Qe_3$, and $\text{curl} \, \mathbf{u} = \partial Q / \partial x_2 - \partial P / \partial x_3$. \hfill \square
From (18), after use of the Green theorem, we obtain

\[ M = \mu \bar{\theta} \left\{ J_0 - \int_A \int \left[ \left( \frac{\partial \Phi}{\partial x_2} \right)^2 + \left( \frac{\partial \Phi}{\partial x_3} \right)^2 \right] dA \right\}, \]

where

\[ J_0 = \int_A \int (x_2^2 + x_3^2) dA \]

is the polar moment of inertia of the cross-section for the point \( O \). We write (19) as

\[ M = \bar{\theta} D; \quad D = \mu \left\{ J_0 - \int_A \int \left[ \left( \frac{\partial \Phi}{\partial x_2} \right)^2 + \left( \frac{\partial \Phi}{\partial x_3} \right)^2 \right] dA \right\}, \]

where \( D \) is the torsional rigidity of the rod.

From the analysis presented so far it follows that the torsion problem for a rod with an arbitrary cross-section is solved once we determine the torsion function \( \Phi \). We show next that the stresses and torsion moment are independent of the choice of point \( O \) for the origin of the coordinate system \( \bar{x}_i, i = 1, 2, 3 \) (see Fig. 3). Suppose we take a point \( O_1 \) with coordinates \( x_1(0), x_2(0), x_3(0) \) for the origin. We again use \( x_1, x_2, \) and \( x_3 \) for the coordinates on an arbitrary point in the new coordinate system as with the origin at \( O_1 \). Then axis \( \bar{x}_3 \) is still the axis of rotation, so that instead of (3) we have

\[ u_1^{(1)} = u_1^{(1)}(x_2, x_3), \quad u_2^{(1)} = -\bar{\theta} x_1(x_3 - x_3^{(1)}), \quad u_3^{(1)} = \bar{\theta} x_1(x_2 - x_2^{(1)}), \]

and the components of the stress tensor (instead of (5)) become

\[ \sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_{23} = 0; \]

\[ \sigma_{12} = \mu \left( -\bar{\theta}(x_3 - x_3^{(1)}) + \frac{\partial u_1^{(1)}}{\partial x_2} \right); \quad \sigma_{13} = \mu \left( \bar{\theta}(x_2 - x_2^{(1)}) + \frac{\partial u_1^{(1)}}{\partial x_3} \right). \]

By using the same procedure as before we obtain that \( \Phi^{(1)} \) defined by

\[ u_1^{(1)} = \bar{\theta} \Phi^{(1)}(x_2, x_3), \]

satisfies

\[ \nabla^2 \Phi^{(1)}(x_2, x_3) = 0, \]

subject to

\[ \frac{d}{dn}(\Phi^{(1)} + x_3^{(1)} x_2 - x_2^{(1)} x_3) = x_3 n_2 - x_2 n_3 \quad \text{on } C. \]

Note that \( \Phi^{(1)} + x_3^{(1)} x_2 - x_2^{(1)} x_3 \) is harmonic, if \( \Phi^{(1)} \) is harmonic. Since \( \Phi^{(1)} + x_3^{(1)} x_2 - x_2^{(1)} x_3 \) and \( \Phi \) satisfy the same boundary conditions it follows from the uniqueness theorem proved earlier in this section (see (17)) that \( \Phi^{(1)} + x_3^{(1)} x_2 - x_2^{(1)} x_3 \) and \( \Phi \) differ at most by a constant. Thus,

\[ \Phi^{(1)} = \Phi - x_3^{(1)} x_2 + x_2^{(1)} x_3 + c, \]
where \( c \) is a constant. By using (22) it follows that the system of stresses obtained by using \( \Phi^{(1)} \) is identical with the system of stresses obtained by using \( \Phi \). As far as displacements are concerned, we obtain from (21) and (3),
\[
\begin{align*}
    u_2^{(1)} &= u_2 + \bar{\theta}x_1x_3^{(1)}; \\
    u_3^{(1)} &= u_3 - \bar{\theta}x_1x_2^{(1)}.
\end{align*}
\]  
(26)

From (22) and the fact the stresses are the same it follows that
\[
\begin{align*}
    \sigma_{12} &= \mu \left( -\bar{\theta}(x_3 - x_3^{(1)}) + \frac{\partial u_1^{(1)}}{\partial x_2} \right) = \mu \left( -\bar{\theta}x_3 + \frac{\partial u_1^{(1)}}{\partial x_2} \right); \\
    \sigma_{13} &= \mu \left( \bar{\theta}(x_2 - x_3^{(1)}) + \frac{\partial u_1^{(1)}}{\partial x_3} \right) = \mu \left( \bar{\theta}x_2 + \frac{\partial u_1^{(1)}}{\partial x_3} \right),
\end{align*}
\]  
(27)
or
\[
\begin{align*}
    \frac{\partial u_1^{(1)}}{\partial x_2} &= \frac{\partial u_1^{(1)}}{\partial x_2} - \bar{\theta}x_3^{(1)}; \\
    \frac{\partial u_1^{(1)}}{\partial x_3} &= \frac{\partial u_1^{(1)}}{\partial x_3} + \bar{\theta}x_2^{(1)}.
\end{align*}
\]  
(28)

From (28) it follows that
\[
    u_1^{(1)} = u_1 - \bar{\theta}x_3^{(1)}x_2 + f(x_3),
\]  
(29)

where
\[
f(x_3) = \bar{\theta}x_2^{(1)} + d,
\]  
(30)

and \( d \) is a constant. By using (26),(29) we conclude that the displacement fields corresponding to two choices of the origin of the coordinate system differ for a rigid body displacement.

We now discuss a different approach to the torsion problem. This approach was presented by Prandtl. The equilibrium equations (6) could be satisfied if we introduce the Prandtl stress function \( \Psi(x_2, x_3) \) by the relations
\[
\begin{align*}
    \sigma_{12} &= \mu \bar{\theta} \frac{\partial \Psi}{\partial x_3}; \\
    \sigma_{13} &= -\mu \bar{\theta} \frac{\partial \Psi}{\partial x_2}.
\end{align*}
\]  
(31)

If we use (31) in (5) we obtain
\[
\begin{align*}
    \mu \bar{\theta} \frac{\partial \Psi}{\partial x_3} &= \mu \left( -\bar{\theta}x_3 + \frac{\partial u_1}{\partial x_2} \right); \\
    -\mu \bar{\theta} \frac{\partial \Psi}{\partial x_2} &= \mu \left( \bar{\theta}x_2 + \frac{\partial u_1}{\partial x_3} \right).
\end{align*}
\]  
(32)

By differentiating (32)_1 with respect to \( x_3 \) and (32)_2 with respect to \( x_2 \) and by subtracting the results, we obtain
\[
\nabla^2 \Psi = \frac{\partial^2 \Psi}{\partial x_2^2} + \frac{\partial^2 \Psi}{\partial x_3^2} = -2.
\]  
(33)

\(^3\)Prandtl introduced the method in his 1903 paper (see Prandtl (1903)). Prandtl's stress function is a special case of the Finzi stress functions (see (4.5-29)).
Equation (33) is Poisson’s differential equation. To obtain the boundary conditions corresponding to (33) we use (31) in (10) so that

\[
\sigma_{12} \frac{dx_3}{dS} - \sigma_{13} \frac{dx_2}{dS} = \mu \vec{\theta} \left( \frac{\partial \Psi}{\partial x_3} \frac{dx_3}{dS} + \frac{\partial \Psi}{\partial x_2} \frac{dx_2}{dS} \right)
\]

\[
= \mu \vec{\theta} \frac{d\Psi}{dS} = 0, \quad \text{on } C. 
\]  

(34)

From (34) follows

\[
\Psi = \text{const.} \quad \text{on } C.  
\]  

(35)

We return later to the problem of specifying the constant in (35). In a simply connected region (cross-section without holes) we may take \( \Psi = 0 \) on \( C \). Note that (33) subject to (35) constitutes the \textit{Dirichlet} type of boundary value problem.

We determine the resultant moment of external forces applied at the ends of the rod. By using (18) and (31) it follows that

\[
M = \int_A \int (\sigma_{13}x_2 - \sigma_{12}x_3) \, dA = -\mu \vec{\theta} \int_A \int \left( \frac{\partial \Psi}{\partial x_2} x_2 + \frac{\partial \Psi}{\partial x_3} x_3 \right) \, dx_2 dx_3
\]

\[
= -\mu \vec{\theta} \int_A \int \left( \frac{\partial}{\partial x_2} (\Psi x_2) + \frac{\partial}{\partial x_3} (\Psi x_3) \right) \, dx_2 dx_3 + 2\mu \vec{\theta} \int_A \int \Psi \, dx_2 dx_3. 
\]  

(36)

Now from Gauss’s theorem and the boundary condition \( \Psi = 0 \) on \( C \) we obtain

\[
M = 2\mu \vec{\theta} \int_A \int \Psi \, dA.  
\]  

(37)

To determine the resultant force, we calculate its components along the \( \bar{x}_2 \) and \( \bar{x}_3 \) axes. Thus

\[
Q_2 = \int_A \int \sigma_{12} \, dA = \mu \vec{\theta} \int_A \int \frac{\partial \Psi}{\partial x_3} \, dA = -\mu \vec{\theta} \int_C \Psi \, dx_2 = 0,  
\]  

(38)

where we used Gauss’s theorem. Similarly we can show that

\[
Q_3 = \int_A \int \sigma_{13} \, dA = 0,  
\]  

(39)

so that \textit{the resultant force acting on each base of the rod is equal to zero}.

There exists a connection between the torsion function introduced by (7) and the Prandtl stress function introduced by (31). Namely, by substituting (7) in (5) and comparing it with (31) we conclude that

\[
-x_3 + \frac{\partial \Phi}{\partial x_2} = \frac{\partial \Psi}{\partial x_3}; \quad x_2 + \frac{\partial \Phi}{\partial x_3} = -\frac{\partial \Psi}{\partial x_2}.  
\]  

(40)
Since $\Phi$ is a harmonic function there exists a conjugate function $\bar{\Phi}$ connected with $\Phi$ by the Cauchy-Riemann equations (see Lavrenteev and Shabat (1987))

$$\frac{\partial \Phi}{\partial x_2} = \frac{\partial \bar{\Phi}}{\partial x_3}; \quad \frac{\partial \Phi}{\partial x_3} = -\frac{\partial \bar{\Phi}}{\partial x_2}. \quad (41)$$

Using (41) in (40) we obtain

$$\frac{\partial \Psi}{\partial x_2} = \frac{\partial \bar{\Phi}}{\partial x_2} - x_2; \quad \frac{\partial \Psi}{\partial x_3} = \frac{\partial \bar{\Phi}}{\partial x_3} - x_3. \quad (42)$$

By integrating (42) it follows that

$$\Psi = \bar{\Phi} - \frac{x_2^2 + x_3^2}{2} + c, \quad (43)$$

where $c$ is a constant. From (43) and the boundary conditions on $\Psi$ we conclude that the torsion problem is solved if we find a harmonic function $\bar{\Phi}$, that is,

$$\frac{\partial^2 \bar{\Phi}}{\partial x_2^2} + \frac{\partial^2 \bar{\Phi}}{\partial x_3^2} = 0, \quad (44)$$

satisfying

$$\bar{\Phi} = \frac{x_2^2 + x_3^2}{2} + c \quad \text{on } C. \quad (45)$$

Thus, with the conjugate function, the torsion problem is reduced to the Dirichlet type of boundary value problem.

We now check the compatibility conditions. Since $\Theta = \sigma_{ii} = 0$ and body forces are absent (i.e., $f_i = 0$) the conditions (3.7-21) become

$$\nabla^2 \sigma_{12} = 0; \quad \nabla^2 \sigma_{13} = 0. \quad (46)$$

By using (31) in (46), we obtain, for example,

$$\nabla^2 \mu \frac{\partial \Psi}{\partial x_3} = \mu \frac{\partial}{\partial x_3} (\nabla^2 \Psi) = 0. \quad (47)$$

Thus, (46)_1 holds. Similarly it could be shown that (46)_2 holds. Therefore, the compatibility equations are satisfied.

We now examine some properties of the obtained solution. First we show that the curves

$$\Psi(x_2, x_3) = \text{const.}, \quad (48)$$

have an interesting physical interpretation. From (48) it follows that

$$\frac{\partial \Psi}{\partial S} = \frac{\partial \Psi}{\partial x_2} \frac{\partial x_2}{\partial S} + \frac{\partial \Psi}{\partial x_3} \frac{\partial x_3}{\partial S} = 0. \quad (49)$$

Since (see Fig. 5)

$$\frac{\partial x_2}{\partial S} = -n_3; \quad \frac{\partial x_3}{\partial S} = n_2, \quad (50)$$
it follows that (49), (50), and (31) lead to

\[ \sigma_{12} n_2 + \sigma_{13} n_3 = 0. \] (51)

From (51) we conclude that the stress vector has the direction of the tangent to the curve \( \Psi = \text{const.} \). The curves \( \Psi = \text{const.} \) are called the curves of shearing stress. Since on \( C \) we have \( \Psi = 0 \), it follows that \( C \) is also a curve of the shearing stress. The stress vector for the unit normal to \( \Psi = \text{const.} \) is

\[ \mathbf{p}_\Psi = \sigma_{12} \mathbf{e}_2 + \sigma_{13} \mathbf{e}_3. \] (52)

By using (31) its magnitude becomes

\[ |\mathbf{p}_\Psi| = \sqrt{\sigma_{12}^2 + \sigma_{13}^2} = \mu \sqrt{\left( \frac{\partial \Psi}{\partial x_2} \right)^2 + \left( \frac{\partial \Psi}{\partial x_3} \right)^2}. \] (53)

Since both components of the stress tensor \( \sigma_{12} \) and \( \sigma_{13} \) are harmonic functions (see (46)) then \( \sigma_{12} \) and \( \sigma_{13} \) attain their maximum on the boundary \( C \) of the cross-section \( A \) (see Tychonoff and Samarski (1959)). To show that the maximum of \( |\mathbf{p}_\Psi| = \sqrt{\sigma_{12}^2 + \sigma_{13}^2} \) is on the boundary \( C \) of \( A \) we may proceed as follows. Assume the contrary: suppose there exists a point \( M \in A \) such that \( |\mathbf{p}_\Psi| \) attains maximum for some \( \Psi \). We choose a new coordinate system inside \( \Psi \) with one of the axes parallel to \( \bar{x}_1 \), one parallel to \( \mathbf{p}_\Psi \) at \( M \), and the third orthogonal to them. In this coordinate system only one component of \( \mathbf{p}_\Psi \), say \( \sigma'_{12} \), is different from zero. Since \( \sigma'_{12} \) is a harmonic (it must satisfy the compatibility equations) function it, contrary to our assumption, attains its maximum on \( C \) and not inside \( A \).

We turn now to the rods with multiple connected cross-section. We consider a body bounded by several cylindrical surfaces with parallel axes. In Fig. 6 we show a cross-section of such a rod.

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\( \text{Figure 6} \)

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\( ^4 \)The first proof that the maximal shear stress is at the boundary of the cross-section was given by Pólya (1930).
5.5 Torsion of a prismatic rod with an arbitrary cross-section

Instead of (35) we now have

$$\Psi = a_i; \quad \text{on} \ C_i; \quad i = 1, 2, 3, \ldots, n. \quad (54)$$

The components of the resultant force at the bases of the rod are (see (38))

$$Q_2 = \int_A \int_A \sigma_{12}dA = \mu \bar{\theta} \int_A \int_A \frac{\partial \Psi}{\partial x_3}dA = -\mu \bar{\theta} \int_C \Psi dx_2 = \mu \bar{\theta} \int_C \Psi n_3dS;$$

$$Q_3 = \int_A \int_A \sigma_{13}dA = -\mu \bar{\theta} \int_A \int_A \frac{\partial \Psi}{\partial x_2}dA = -\mu \bar{\theta} \int_C \Psi dx_3 = -\mu \bar{\theta} \int_C \Psi n_2dS, \quad (55)$$

where $C$ denotes the contour $C = C_0 + C_1 + \ldots C_n$. From (55) it follows that

$$Q_2 = \mu \bar{\theta} \sum_{i=0}^{n} \int_{C_i} \Psi n_3dS = \mu \bar{\theta} \sum_{i=0}^{n} a_i \int_{C_i} n_3dS;$$

$$Q_3 = -\mu \bar{\theta} \sum_{i=0}^{n} \int_{C_i} \Psi n_2dS = -\mu \bar{\theta} \sum_{i=0}^{n} a_i \int_{C_i} n_2dS. \quad (56)$$

However

$$\int_{C_i} n_3dS = -\int_{C_i} dx_2 = 0; \quad \int_{C_i} n_2dS = \int_{C_i} dx_3 = 0. \quad (57)$$

so that

$$Q_2 = 0, \quad Q_3 = 0. \quad (58)$$

The resultant couple is

$$M = \int_A \int_A \left( \sigma_{13}x_2 - \sigma_{12}x_3 \right)dA = -\mu \bar{\theta} \int_A \int_A \left( \frac{\partial \Psi}{\partial x_2}x_2 + \frac{\partial \Psi}{\partial x_3}x_3 \right)dx_2dx_3$$

$$= -\mu \bar{\theta} \int_A \int_A \left( \frac{\partial}{\partial x_2} \Psi x_2 + \frac{\partial}{\partial x_3} \Psi x_3 \right)dx_2dx_3 + 2\mu \bar{\theta} \int_A \int_A \Psi dx_2dx_3$$

$$= -\mu \bar{\theta} \left[ \sum_{i=0}^{n} \int_{C_i} \Psi (x_2n_2 + x_3n_3)dS \right] + 2\mu \bar{\theta} \int_A \int_A \Psi dA. \quad (59)$$

By using (54) in (59) and remembering that the outer contour is described in the counter-clockwise and all inner contours in the clockwise directions, we obtain

$$M = 2\mu \bar{\theta} \left[ \sum_{i=1}^{n} a_iA_i - a_0A_0 \right] + 2\mu \bar{\theta} \int_A \int_A \Psi dA, \quad (60)$$

where $A_i, i = 0, 1, 2, \ldots, n$ are areas bounded by $C_i$. 
Finally we present the so-called *circulation theorem*. Namely, consider a closed curve \( \psi \) within \( A \). Let \( n \) be the unit normal and \( t \) the unit tangent of \( \psi \). The integral of the stress vector \( p_n \) calculated along \( \psi \), that is,

\[
\Gamma = \oint_\psi [\sigma_{12}e_2 + \sigma_{13}e_3] \cdot t dS,
\]

is called the *circulation of \( p_n \)*. The unit tangent is

\[
t_2 = \cos \angle(t, \bar{x}_2); \quad t_3 = \cos \angle(t, \bar{x}_3).
\]

Therefore (see Fig. 5)

\[
t_2 = \frac{dx_2}{dS}; \quad t_3 = \frac{dx_3}{dS}.
\]

By using (5) and (63) in (61) we obtain

\[
\Gamma = \oint_\psi \left[ \mu \left( -\partial x_3 + \frac{\partial u_1}{\partial x_2} \right) dx_2 + \mu \left( \partial x_2 + \frac{\partial u_1}{\partial x_3} \right) dx_3 \right],
\]

or

\[
\Gamma = \mu \bar{\theta} \oint_\psi [-x_3 dx_2 + x_2 dx_3] + \mu \oint_\psi \left[ \frac{\partial u_1}{\partial x_2} dx_2 + \frac{\partial u_1}{\partial x_3} dx_3 \right].
\]

The second term in (65) is zero since \( u_1 \) is a single-valued function. The first term is

\[
\mu \bar{\theta} \oint_\psi [-x_3 dx_2 + x_2 dx_3] = 2\mu \bar{\theta} A_\psi,
\]

where \( A_\psi \) is the area enclosed by \( \psi \). Therefore (65) becomes

\[
\Gamma = 2\mu \bar{\theta} A_\psi.
\]

Expression (67) is *Bredt’s theorem: circulation of the shear stress \( \tau = p_n \cdot t \) along any closed contour completely inside the cross-section of the rod, when the rod is in torsion, is equal to the area enclosed by the contour multiplied by \( 2\bar{\theta}\mu \).*

We turn now to the solution of torsion problems with some specific cross-sections.

**i) Rod with circular cross-section**

Consider a rod with circular cross-section. Let \( R \) be the radius of the rod. Then we have (see Fig. 7)

\[
x_3 n_2 - x_2 n_3 = 0,
\]

so that solution to the boundary value problem (9),(14) reads

\[
\Phi = 0.
\]
From (69) we conclude that by torsion of a rod with circular cross-section there is no warping (deplanation) of the cross-section.

In the elementary approach (strength of materials) this is taken as the hypothesis (the Bernoulli hypothesis). From (19) it follows that

\[ M = \mu \bar{\theta} J_0, \tag{70} \]

where \( J_0 = \pi R^4 / 2 \), whereas (5) leads to

\[ \sigma_{12} = -\mu \bar{\theta} x_3; \quad \sigma_{13} = \mu \bar{\theta} x_2. \tag{71} \]

The shear stress, determined from (71) becomes

\[ |\mathbf{p}_\psi| = p_\psi = \sqrt{\sigma_{12}^2 + \sigma_{13}^2} = \mu \bar{\theta} \rho, \tag{72} \]

where \( \rho = \sqrt{x_2^2 + x_3^2} \). Equation (72) is recognized as a well-known relation from the elementary approach to the torsion problem.

The interesting property of (72) is that for \( \rho = R \) we have \( p_\psi = \text{const} \). By considering equation (33) subject to

\[ \Psi = 0, \quad \mu \bar{\theta} \frac{\partial \Psi}{\partial n} = \mu \bar{\theta} \sqrt{\sigma_{12}^2 + \sigma_{13}^2} = \text{const. on } C, \tag{73} \]

Serrin (1971) showed that: the only cross-section for which in torsion the shear stress is constant on the boundary \( C \) is circular cross-section.

ii) Rod with elliptical cross-section

Suppose that the cross-section is bounded by the ellipse described by

\[ f(x_2, x_3) = \frac{x_2^2}{a^2} + \frac{x_3^2}{b^2} - 1, \tag{74} \]
where \( a \) and \( b \) are constants. We use the Prandtl stress function method. Suppose that \( \Psi = Hf \) where \( H \) is a constant that is determined from the boundary condition \( \Psi = 0 \) on \( C \). Thus \( \Psi \) becomes

\[
\Psi = -\frac{a^2b^2}{a^2 + b^2}\left(\frac{x_2^2}{a^2} + \frac{x_3^2}{b^2} - 1\right). \tag{75}
\]

By using (75) in (37) we obtain

\[
M = 2\mu\frac{a^2b^2}{a^2 + b^2}\int_A\int\left(1 - \frac{x_2^2}{a^2} - \frac{x_3^2}{b^2}\right)dA. \tag{76}
\]

Since the cross-sectional area is \( A = \pi ab \) and

\[
I_2 = \int_A\int x_2^2dA = \frac{1}{4}\pi ab^3, \quad I_3 = \int_A\int x_3^2dA = \frac{1}{4}\pi a^3b, \tag{77}
\]

where \( I_2 \) and \( I_3 \) are axial moments of inertia, we obtain

\[
M = \mu\frac{a^3b^3}{a^2 + b^2}\pi. \tag{78}
\]

The torsional rigidity (see (20)) now becomes

\[
D = \mu\frac{a^3b^3}{a^2 + b^2}\pi. \tag{79}
\]

If we solve (78) for \( M \) and use the result in (75) the Prandtl stress function could be written in the form

\[
\Psi = \frac{M}{\pi ab}\left(1 - \frac{x_2^2}{a^2} - \frac{x_3^2}{b^2}\right)\frac{1}{\mu\bar{\theta}}. \tag{80}
\]

Note that the lines \( \Psi = \text{const.} \) are ellipses. Now from (80) and (31) we obtain

\[
\sigma_{12} = -\frac{2Mx_3}{\pi ab^3}; \quad \sigma_{13} = \frac{2Mx_2}{\pi a^3b}. \tag{81}
\]

As is seen from (81) the ratio \( \sigma_{13}/\sigma_{12} \) is proportional to \( x_2/x_3 \). Therefore \( \sigma_{13}/\sigma_{12} \) is constant along a line \( \bar{\Omega} \) (see Fig. 8). Finally we determine the components of the displacement vector. By using (3) with \( \bar{\theta} \) determined from (78) we have

\[
u_2 = -\frac{M(a^2 + b^2)}{\mu\pi a^3b^3}x_1x_3; \quad u_3 = \frac{M(a^2 + b^2)}{\mu\pi a^3b^3}x_1x_2. \tag{82}
\]

Also from (5) and (81) with \( \bar{\theta} \) determined from (78) we have

\[
\frac{\partial u_1}{\partial x_2} = \frac{b^2 - a^2}{a^2 + b^2}\bar{\theta}x_3; \quad \frac{\partial u_1}{\partial x_3} = \frac{b^2 - a^2}{a^2 + b^2}\bar{\theta}x_2. \tag{83}
\]
5.5 Torsion of a prismatic rod with an arbitrary cross-section

Assuming that the point 0 is fixed and by integrating (83) we finally obtain

\[ u_1 = \frac{b^2 - a^2}{a^2 + b^2} \tilde{\theta} x_2 x_3 = \frac{M(b^2 - a^2)}{\mu \pi a^3 b^3} x_2 x_3. \]  

(84)

As is seen from (84) the plane cross-sections in the undeformed state are not plane in the deformed state. The contour lines \( u_1 = \text{const.} \) are shown in Fig. 8 and represent hyperbolas. The shear stress has the value (see (31) and (80))

\[ p_\psi = \frac{2\mu \tilde{\theta}}{a^2 + b^2} \sqrt{a^4 x_2^2 + b^4 x_2^2}. \]  

(85)

Suppose that \( a > b \). Then the maximal value of \( p_\psi \) is

\[ (p_\psi)_{\text{max}} = \frac{2\mu \tilde{\theta} a^2 b}{a^2 + b^2}, \]  

(86)

for the point with coordinates \( x_2 = 0, x_3 = b \).

iii) Rod with narrow rectangular cross-section

Suppose that the cross-section of the rod is the narrow rectangle shown in Fig. 9. Thus, we assume that \( a \gg b \).
We present an approximate solution by assuming the Prandtl stress function in the form
\[ \Psi = b^2 - x_2^2. \]  
(87)
The function (87) satisfies equation (33) and the boundary condition \( \Psi = 0 \) on the part of the boundary \( x_2 = \pm b \). On the part \( x_3 = \pm a \) the boundary condition \( \Psi = 0 \) is not satisfied. Therefore (87) is an approximate solution of the torsion problem since the boundary conditions are not satisfied exactly. By using (87) the stresses become
\[ \sigma_{12} = 0; \quad \sigma_{13} = 2\mu \bar{\theta}x_2, \]  
(88)
and the couple (37) is
\[ M = \frac{16}{3} \mu \bar{\theta}ab^3. \]  
(89)
The maximal shear stress in this case is
\[ (\sigma_{13})_{\text{max}} = 2\mu \bar{\theta}b = \frac{3M}{8ab^2}. \]  
(90)
iv) Rod with rectangular cross-section

We now treat the case when \( a \) is not much larger than \( b \). In this case we take the Prandtl stress function in the form
\[ \Psi = b^2 - x_2^2 + T(x_2, x_3). \]  
(91)
By substituting (91) into (33) and (35) we obtain that \( T \) satisfies
\[ \nabla^2 T = 0, \]  
(92)
with the boundary conditions
\[ T = 0 \text{ for } x_2 = \pm b; \quad T = x_2^2 - b^2 \text{ for } x_3 = \pm a. \]  
(93)
The solution of (92), (93) we assume in the form
\[ T = f(x_2)g(x_3). \]  
(94)
Then, from (92) it follows that
\[ \frac{d^2 f}{dx_2^2} \frac{1}{f} = -\frac{d^2 g}{dx_3^2} \frac{1}{g}, \]  
(95)
or
\[ f = C_1 \cos(\lambda x_2) + C_2 \sin(\lambda x_2); \quad g = C_3 \text{ch}(\lambda x_3) + C_4 \text{sh}(\lambda x_3), \]  
(96)
where \( \lambda \) and \( C_i, i = 1, \ldots, 4 \) are constants. From (93) we have
\[ \cos(\lambda b) = 0 \Rightarrow \lambda = \frac{\pi n}{2b}, \quad n = 1, 3, 5, \ldots \]  
(97)
Due to the symmetry we conclude that $T$ must be an even function of $x_3$. It follows then that $C_4 = 0$. Therefore $T$ becomes

$$T = \sum_{n=1,3,...}^{\infty} A_n \cos \left( \frac{n\pi x_2}{2b} \right) \operatorname{ch} \left( \frac{n\pi x_3}{2b} \right),$$

(98)

where $A_n$, $n = 1, 3, 5, \ldots$, are constants. The boundary condition (93) leads to

$$\sum_{n=1,3,...}^{\infty} A_n \cos \left( \frac{n\pi x_2}{2b} \right) \operatorname{ch} \left( \frac{n\pi a}{2b} \right) = x_2^2 - b^2.$$  

(99)

We now expand the function $x_2^2 - b^2$ in a Fourier series. Since it is an even function it has only the "cos" terms. To obtain coefficients in the series, we multiply (9) by ($m\pi x_2/2b$) and integrate between $x_2 = -b$ and $x_2 = b$. The result is

$$\sum_{n=1,3,...}^{\infty} \int_{-b}^{b} A_n \cos \left( \frac{n\pi x_2}{2b} \right) \operatorname{ch} \left( \frac{n\pi a}{2b} \right) \cos \left( \frac{m\pi x_2}{2b} \right) dx_2
= \int_{-b}^{b} (x_2^2 - b^2) \cos \left( \frac{m\pi x_2}{2b} \right) dx_2. 

(100)

The left-hand side of (100) is different from zero only when $m = n$, so that

$$A_n \operatorname{ch} \left( \frac{n\pi a}{2b} \right) = \frac{1}{b} \int_{-b}^{b} (x_2^2 - b^2) \cos \left( \frac{n\pi x_2}{2b} \right) dx_2. 

(101)

From (101) we obtain the Fourier coefficients as

$$A_n = (-1)^{(n+1)/2} \frac{32b^2}{\pi^3 n^3} \frac{\operatorname{ch} \left( \frac{n\pi a}{2b} \right)}{\operatorname{ch} \left( \frac{n\pi a}{2b} \right)}, \quad n = 1, 3, 5, \ldots, 

(102)

where we used $\sin(n\pi/2) = -(-1)^{(n+1)/2}$ for $n = 1, 3, 5, \ldots$. From (1), (98), and (102) the Prandtl stress function could be expressed in two equivalent forms

$$\Psi = b^2 - x_2^2 + \frac{32b^2}{\pi^3} \sum_{n=1,3,...}^{\infty} \frac{(-1)^{(n+1)/2} \cos \left( \frac{n\pi x_2}{2b} \right) \operatorname{ch} \left( \frac{n\pi x_3}{2b} \right)}{n^3} 
\times \frac{\operatorname{ch} \left( \frac{n\pi a}{2b} \right)}{\operatorname{ch} \left( \frac{n\pi a}{2b} \right)} 

(103)$$

$$= b^2 - x_2^2 + \frac{32b^2}{\pi^3} \sum_{m=0,1,2,...}^{\infty} \frac{(-1)^{m+1}}{(2m+1)^3} 
\times \frac{\cos \left( \frac{(2m+1)\pi x_2}{2b} \right) \operatorname{ch} \left( \frac{(2m+1)\pi x_3}{2b} \right)}{\operatorname{ch} \left( \frac{(2m+1)\pi a}{2b} \right)}.$$
By using (31) and (103) we obtain the stresses as

\[ \sigma_{12} = \frac{16 \mu \bar{\theta}}{\pi^2} \frac{b}{\pi^2} \sum_{n=1,3,\ldots}^\infty \frac{(-1)^{(n+1)/2} \cos \left( \frac{n \pi x_2}{2b} \right)}{n^2} \frac{\sinh \left( \frac{n \pi x_3}{2b} \right)}{\cosh \left( \frac{n \pi a}{2b} \right)}; \]

\[ \sigma_{13} = -\frac{16 \mu \bar{\theta} \partial \Psi}{\partial x_2} \]

\[ = \frac{2x_2 - 16b}{\pi^2} \sum_{n=1,3,\ldots}^\infty \frac{(-1)^{(n+1)/2}}{n^2} \frac{\sin \left( \frac{n \pi x_2}{2b} \right)}{\cosh \left( \frac{n \pi a}{2b} \right)} \] \quad (104)

The maximal shear stress for the case when \( a > b \) is at the midpoint of the longer side, that is, for \( x_3 = 0, x_2 = b \). Thus, from (104) we have

\[ |\sigma_{13}|_{\text{max}} = 2 \mu \bar{\theta} \left[ 1 - \frac{8}{\pi^2} \sum_{n=1,3,\ldots}^\infty \frac{1}{n^2 \cosh \left( \frac{n \pi a}{2b} \right)} \right]. \quad (105) \]

Also by using (104) in (37) we get

\[ M = \frac{1}{3} \mu \bar{\theta} (2a)(2b)^3 \left[ 1 - \frac{192b}{\pi^5 a} \sum_{n=1,3,\ldots}^\infty \frac{\tanh \left( \frac{n \pi a}{2b} \right)}{n^5} \right]. \quad (106) \]

For the case when the rectangle is thin, that is, \( a >> b \), the equation (106) gives

\[ M = \frac{1}{3} \mu \bar{\theta} (2a)(2b)^3 \left[ 1 - 0.63 \frac{b}{a} \right]. \quad (107) \]

This is an improved result when compared to (89). Finally the \( u_1 \) component of the displacement vector is determined from (104) and (5). The result is

\[ u_1 = \bar{\theta} \left[ x_2 x_3 + \sum_{n=1,3,\ldots}^\infty \frac{(-1)^{(n+1)/2}}{n^3} \frac{\sin \left( \frac{n \pi x_2}{2b} \right)}{\cosh \left( \frac{n \pi a}{2b} \right)} \right]. \quad (108) \]

v) **Hollow rod with elliptical cross-section**

Consider a rod bounded by similar elliptical cylinders. The boundary of the outer cylinder is described by

\[ \frac{x_2^2}{a^2} + \frac{x_3^2}{b^2} = 1. \quad (109) \]
5.5 Torsion of a prismatic rod with an arbitrary cross-section

The boundary of the inner cylinder we take as

\[
\frac{x_2^2}{a^2} + \frac{x_3^2}{b^2} = (1 - k)^2; \quad 0 < k < 1. \tag{110}
\]

In this case we take the Prandtl stress function in the form (75), that is,

\[
\Psi = -\frac{a^2 b^2}{a^2 + b^2} \left( \frac{x_2^2}{a^2} + \frac{x_3^2}{b^2} - 1 \right). \tag{111}
\]

The function (111) satisfies (33) and (54) on \( C_0 \), that is, on the outer boundary given by (109). We have to satisfy the boundary condition on the inner boundary

\[
\Psi = a_1; \quad \text{on} \ C_1, \tag{112}
\]

where \( C_1 \) is given by (110). This is easy in the present case since the lines \( \Psi = \text{const.} \) with \( \Psi \) given by (111) are ellipses. Therefore on \( C_1 \) the value of \( \Psi \) is

\[
\Psi = a_1 = \frac{a^2 b^2}{a^2 + b^2} [1 - (1 - k)^2]. \tag{113}
\]

Thus the function (111) represents Prandtl’s function for hollow rod too. The resultant moment could be determined from (60) by taking into account (111) and (113).

vi) Rod with the cross-section in the form of half a ring

Consider a rod with a cross-section shown in Fig. 10. We write the differential equation for the Prandtl stress function in a cylindrical coordinate system as

\[
\frac{\partial^2 \Psi}{\partial r^2} + \frac{1}{r} \frac{\partial \Psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Psi}{\partial \theta^2} = -2. \tag{114}
\]

\[
\text{Figure 10}
\]

Then next expand the right-hand side of (114) into a Fourier series for \( \theta \in [0, \pi] \) to obtain

\[
2 = \frac{8}{\pi} \sum_{n=0}^{\infty} \frac{1}{2n + 1} \sin(2n + 1)\theta. \tag{115}
\]
Next we assume the solution of (114) as

$$\Psi(r, \theta) = \sum_{n=0}^{\infty} f_n(r) \sin(2n + 1) \theta,$$  \hspace{1cm} (116)

where $f_n$ are functions to be determined. By substituting (115) and (116) into (114) we obtain

$$\frac{d^2 f_n}{dr^2} + \frac{1}{r} \frac{df_n}{dr} - \frac{(2n + 1)^2}{r^2} f_n = -\frac{8}{\pi} \frac{1}{2n + 1},$$  \hspace{1cm} (117)

The solution of (117) reads

$$f_n = A_n r^{2n+1} + B_n r^{-2n-1} + \frac{8r^2}{\pi(2n-1)(2n+1)(2n+3)},$$  \hspace{1cm} (118)

where $A_n$ and $B_n$ are arbitrary constants. The function (116) satisfies the boundary condition $\Psi = 0$ for $\theta = 0$ and $\theta = \pi$. To satisfy $\Psi = 0$ for $r = r_1$ and $r = r_2$ we require that

$$f_n(r_1) = f_n(r_2) = 0.$$  \hspace{1cm} (119)

By using (119) in (118) we obtain

$$f_n = \frac{8r_2^2}{\pi(2n-1)(2n+1)(2n+3)} \left[ \rho^2 - a_n \rho^{2n+1} - b_n \rho^{-2n-1} \right],$$  \hspace{1cm} (120)

where

$$\rho = \frac{r}{r_2}; \quad k = \frac{r_1}{r_2}; \quad a_n = \frac{1 - k^{2n+3}}{1 - k^{4n+2}}; \quad b_n = k^{2n+3} \frac{1 - k^{2n-1}}{1 - k^{4n+2}}.$$  \hspace{1cm} (121)

With (120) and (116) the nonzero components of the stress tensor become at an arbitrary point $P$

$$\sigma_{rz} = \mu \bar{\theta} \frac{1}{r} \frac{\partial \Psi}{\partial \theta} = \mu \bar{\theta} r_2 \sum_{n=0}^{\infty} \frac{8}{\pi(2n-1)(2n+1)(2n+3)} (2n+1) \times \left[ \rho - a_n \rho^{2n} - b_n \rho^{-2(n+1)} \right] \cos(2n + 1) \theta;$$

$$\sigma_{\theta z} = \mu \bar{\theta} \frac{\partial \Psi}{\partial r} = -\mu \bar{\theta} r_2 \sum_{n=0}^{\infty} \frac{8}{\pi(2n-1)(2n+1)(2n+3)} (2n+1) \times \left[ \frac{2}{2n+1} \rho - a_n \rho^{2n} + b_n \rho^{-2(n+1)} \right] \sin(2n + 1) \theta.$$  \hspace{1cm} (122)

The maximal value of the shear stress is at the point $r = r_2$ (i.e., $\rho = 1$) and $\theta = \pi/2$. 
vii) Prismatic rectangular rod with a crack

Consider a rod with the cross-section shown in Fig. 11. It has a rectangular crack along the line 3’ - 2’. The length of the crack is \( h \). We look for the solution of equation (5.5-33); that is,

\[
\nabla^2 \Psi = \frac{\partial^2 \Psi}{\partial x_2^2} + \frac{\partial^2 \Psi}{\partial x_3^2} = -2,
\]

in the domain shown in Fig. 11.

Let \( \Psi_1 \) be the value of \( \Psi \) in the part of the domain 1-1-2-2. We take \( \Psi_1 \) in the form (see (5.5-91))

\[
\Psi_1 = a^2 - x_3^2 + \sum_{n=1,3,\ldots}^{\infty} \left( A_n \text{ch} \left( \frac{n\pi x_2}{2a} \right) + B_n \text{sh} \left( \frac{n\pi x_2}{2a} \right) \right) \cos \left( \frac{n\pi x_3}{2a} \right)
\]

\[
= \sum_{n=1,3,\ldots}^{\infty} \left( \frac{32a^2}{n^3\pi^3} (-1)^{(n-1)/2} + A_n \text{ch} \left( \frac{n\pi x_2}{2a} \right) \right)
\]

\[
+ B_n \text{sh} \left( \frac{n\pi x_2}{2a} \right) \cos \left( \frac{n\pi x_3}{2a} \right),
\]

(124)

where we (as in Section 5.5 used the Fourier series expansion of the function \( a^2 - x_2^2 \)). The function (124) satisfies (123) in 1-1-2-2 and the boundary condition \( \Psi_1 = 0 \) along the edge 1-2; that is, \( x_3 = \pm a \). To satisfy the boundary condition on the line 1-1, that is, for \( x_2 = b \), we must have

\[
\frac{32a^2}{n^3\pi^3} (-1)^{(n-1)/2} + A_n \text{ch} \left( \frac{n\pi b}{2a} \right) + B_n \text{sh} \left( \frac{n\pi b}{2a} \right) = 0.
\]

(125)

By solving (125) for \( A_n \) we obtain

\[
A_n = -\frac{32a^2}{n^3\pi^3} (-1)^{(n-1)/2} \frac{1}{\text{ch} \left( \frac{n\pi b}{2a} \right)} - B_n \text{th} \left( \frac{n\pi b}{2a} \right).
\]

(126)
Let $\Psi_2$ be the value of $\Psi$ in the part of the domain $2-2'-3'-3$. We assume $\Psi_2$ in the form

$$
\Psi_2 = \left( ax_3 - x_3^2 \right) + \sum_{m=1,2,\ldots}^{\infty} \left( C_m \text{ch} \left( \frac{m\pi x_2}{a} \right) + D_m \text{sh} \left( \frac{m\pi x_2}{a} \right) \right) \sin \left( \frac{m\pi x_3}{a} \right)
$$

(127)

where we used the Fourier sine series expansion for $(x_2 a - x_2^2)$. The function (127) satisfies (123) and the boundary condition $\Psi_2 = 0$ along the edges 2-3 and 2'-3'. To satisfy the boundary condition along 3-3 we must have $\Psi_2 = 0$ for $x_2 = -h$. Thus,

$$
\frac{4a^2}{m^3\pi^3} [1 - (-1)^m] + C_m \text{ch} \left( \frac{m\pi h}{a} \right) - D_m \text{sh} \left( \frac{m\pi h}{a} \right) = 0,
$$

(128)

or

$$
C_m = -\frac{4a^2}{m^3\pi^3} [1 - (-1)^m] \frac{1}{\text{ch} \left( \frac{m\pi h}{a} \right)} + D_m \text{th} \left( \frac{m\pi h}{a} \right).
$$

(129)

We turn now to the matching conditions along the line 2-2'-2. We must have

$$
\Psi_1(x_2 = 0, x_3) = \Psi_2(x_2 = 0, x_3);
$$

(130)

To satisfy (130) we first expand the function

$$
f(x_3) = \begin{cases} 
\sin \frac{m\pi x_3}{a}, & -a < x_3 < 0 \\
-\sin \frac{m\pi x_3}{a}, & 0 < x_3 < a
\end{cases}
$$

(131)

into a Fourier cosine series as

$$
f_1(x_3) = -(-1)^m \frac{2m}{\pi} \sum_{n=1,3,\ldots}^{\infty} \frac{1}{m^2 - \left(\frac{n}{2}\right)^2} \cos \left( \frac{n\pi x_3}{2a} \right).
$$

(132)
By using (132) in (127) the condition (130) becomes

\[
\sum_{n=1,3,\ldots}^{\infty} \left( \frac{32a^2}{n^3\pi^3} (-1)^{(n-1)/2} + A_n \right) \cos \left( \frac{n\pi x_3}{2a} \right) = - \sum_{m=1,2,\ldots}^{\infty} \left( \frac{4a^2}{m^3\pi^3} [1 - (-1)^{m}] + C_m \right) (-1)^{m} \frac{2m}{\pi} \sum_{n=1,3,\ldots}^{\infty} \frac{\cos \left( \frac{n\pi x_3}{2a} \right)}{m^2 - \left( \frac{n}{2} \right)^2},
\]

or

\[
A_n = \frac{32a^2}{\pi^3} \left\{ \frac{1}{2\pi} \sum_{m=1,3,\ldots}^{\infty} \frac{1}{m^2 \left[ m^2 - \left( \frac{n}{2} \right)^2 \right]} - \frac{(-1)^{(n-1)/2}}{n^3} \right\}
\]

\[
= - \frac{2}{\pi} \sum_{m=1,2,\ldots}^{\infty} \frac{(-1)^{m} m}{m^2 - \left( \frac{n}{2} \right)^2} C_m.
\]

To obtain equality of the derivatives (130) we calculate

\[
\left( \frac{\partial \Psi_1}{\partial x_2} \right)_{x_2=0} = \sum_{n=1,3,\ldots}^{\infty} \left( B_n \frac{n\pi}{2a} \right) \cos \left( \frac{n\pi x_3}{2a} \right);
\]

\[
\left( \frac{\partial \Psi_2}{\partial x_2} \right)_{x_2=0} = \sum_{m=1,2,\ldots}^{\infty} \left( D_m \frac{m\pi}{a} \right) \sin \left( \frac{m\pi x_3}{a} \right).
\]

Then, we expand the function \( f(x_3) \)

\[
f_2(x_3) = \begin{cases} 
- \cos \frac{n\pi x_3}{2a}, & -a < x_3 < 0 \\
\cos \frac{n\pi x_3}{2a}, & 0 < x_3 < a
\end{cases}
\]

into the Fourier sine series, to obtain

\[
f_2(x_3) = - \frac{2}{\pi} \sum_{m=1,2,\ldots}^{\infty} \frac{(-1)^{m} m}{m^2 - \left( \frac{n}{2} \right)^2} \sin \left( \frac{m\pi x_3}{a} \right).
\]

Finally by using (137) in (135) and (130) we have

\[
\sum_{n=1,3,\ldots}^{\infty} \left( B_n \frac{n\pi}{2a} \right) \left[ \frac{2}{\pi} \sum_{m=1,2,\ldots}^{\infty} \frac{(-1)^{m} m}{m^2 - \left( \frac{n}{2} \right)^2} \sin \left( \frac{m\pi x_3}{a} \right) \right] = \sum_{m=1,2,\ldots}^{\infty} \left( D_m \frac{m\pi}{a} \right) \sin \left( \frac{m\pi x_3}{a} \right).
\]
Solving (138) for \( D_m \) we get
\[
D_m = -\frac{(-1)^{m}}{\pi} \sum_{n=1,3,\ldots}^{\infty} \frac{n}{m^2 - \left(\frac{n}{2}\right)^2} B_n.
\] (139)

By substituting (139) in (129) and the so-obtained result in (134) we get from the condition (126)
\[
B_n + \frac{2}{\pi} \text{ch} \left( \frac{n \pi b}{2a} \right) \sum_{i=1,3,\ldots}^{\infty} i B_i \sum_{m=1,2,\ldots}^{\infty} \frac{m \text{ th} \left( \frac{m \pi h}{a} \right)}{m^2 - \left(\frac{n}{2}\right)^2} \left[ \frac{m^2 - \left(\frac{n}{2}\right)^2}{m^2 - \left(\frac{1}{2}\right)^2} \right] = 0.
\] (140)

With \( B_n \) known from (140) we obtain the solutions \( \Psi_1 \) and \( \Psi_2 \). Then the resultant moment follows as (see (37))
\[
M = 2G\bar{\theta} \left[ \int_{-a}^{a} \left( \int_{0}^{b} \Psi_1 \, dx_2 \right) \, dx_3 + 2 \int_{0}^{a} \left( \int_{0}^{b} \Psi_2 \, dx_2 \right) \, dx_3 \right].
\] (141)

Basilevich (1979) calculated the first five terms in (140) for the case when \( a = b = h = 4 \). He found that for the cross-section without crack, (106) leads to \( M_{wc} = 596\mu\bar{\theta} \), whereas according to (141) the couple is \( M = 349\mu\bar{\theta} \). Thus the ratio
\[
\frac{M}{M_{wc}} = 0.59,
\] (142)
shows the reduction in the couple due to the crack.

### 5.6 Torsion of a rod with variable circular cross-section

We consider a rod with variable cross-section loaded by two couples at its ends (see Fig. 12). Based on the properties of the solution for a rod with constant circular cross-section we assume that in the cylindrical coordinate system \((r, \theta, z)\) the components of the displacement vector are
\[
u_r = u_z = 0; \quad u_\theta = u_\theta(r, z).
\] (1)
5.6 Torsion of a rod with variable circular cross-section

Figure 12

From (1) we obtain the components of the strain tensor as

\[ E_{rr} = E_{\theta\theta} = E_{zz} = E_{rz} = 0; \quad E_{r\theta} = \frac{1}{2} \left( \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right); \quad E_{\theta z} = \frac{1}{2} \frac{\partial u_\theta}{\partial z}, \]  

(2)

so that

\[ \vartheta = \frac{\partial u_r}{\partial r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} + \frac{u_r}{r} = 0. \]  

(3)

The stress tensor is (see (3.6-1))

\[ \sigma_{rr} = \sigma_{\theta\theta} = \sigma_{zz} = \sigma_{rz} = 0; \quad \sigma_{r\theta} = \mu \left( \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right); \quad \sigma_{\theta z} = \mu \frac{\partial u_\theta}{\partial z}. \]  

(4)

With (4) the equilibrium equations (1.5-6) become

\[ \frac{\partial}{\partial r} \left[ \mu \left( \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right) \right] + \frac{\partial}{\partial z} \left[ \mu \frac{\partial u_\theta}{\partial z} \right] = 0, \]  

(5)

or

\[ \frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r} \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2} + \frac{\partial^2 u_\theta}{\partial z^2} = 0. \]  

(6)

Equation (6) could be written as

\[ \frac{\partial}{\partial r} \left[ r^3 \frac{\partial}{\partial r} \left( \frac{u_\theta}{r} \right) \right] + \frac{\partial}{\partial z} \left[ r^3 \frac{\partial}{\partial z} \left( \frac{u_\theta}{r} \right) \right] = 0. \]  

(7)

From (7) we conclude that there exists a function \( F \) such that

\[ \frac{\partial F}{\partial r} = r^3 \frac{\partial}{\partial z} \left( \frac{u_\theta}{r} \right); \quad \frac{\partial F}{\partial z} = -r^3 \frac{\partial}{\partial r} \left( \frac{u_\theta}{r} \right), \]  

(8)

or

\[ \frac{\partial}{\partial z} \left( \frac{u_\theta}{r} \right) = \frac{1}{r^3} \frac{\partial F}{\partial r}; \quad \frac{\partial F}{\partial r} = \frac{1}{r^3} \frac{\partial}{\partial z}. \]  

(9)
Differentiating (9)\_1 with respect to \( r \) and (9)\_2 with respect to \( z \) and by adding the results, we obtain

\[
\frac{\partial^2 F}{\partial r^2} - \frac{3}{r} \frac{\partial F}{\partial r} + \frac{\partial^2 F}{\partial z^2} = 0. \tag{10}
\]

By using (8) and (4) we can express the stress tensor components in terms of \( F \) as

\[
\sigma_{\theta z} = \frac{\mu}{r^2} \frac{\partial F}{\partial r}; \quad \sigma_{r\theta} = -\frac{\mu}{r^2} \frac{\partial F}{\partial z}. \tag{11}
\]

The stress vector on the lateral surface is equal to zero (note that \( n_\theta = 0 \)) so that

\[
\sigma_{\theta z} n_z + \sigma_{r\theta} n_r = 0; \quad \text{on } L. \tag{12}
\]

Since \( n_z = -dr/dS \) and \( n_r = dz/dS \) where \( S \) is the arc-length of the curve \( L \), by substituting (11) into (12) we obtain

\[
\frac{dF}{dS} = 0 \quad \text{on } L. \tag{13}
\]

Therefore \( F \) is determined by solving (10) subject to (13). The resultant couple for any cross-section with the radius \( a \) is determined as

\[
M = \int_0^{2\pi} \int_0^a \sigma_{\theta z} r^2 dr d\theta = 2\pi \int_0^a \sigma_{\theta z} r^2 dr = 2\pi \mu \int_0^a \frac{\partial F}{\partial r} dr = 2\pi \mu [F(a, z) - F(0, z)]. \tag{14}
\]

As an example, consider the case of a conical rod (see Fig. 13). The solution of (10) is

\[
F(r, z) = c \left\{ \frac{z}{(r^2 + z^2)^{1/2}} - \frac{1}{3} \left[ \frac{z}{(r^2 + z^2)^{1/2}} \right]^3 \right\}, \tag{15}
\]

where \( c \) is a constant.

Figure 13
On line \( L \) defined by
\[
    r = (\tan \alpha)z,
\]
we have \( F(r, z) = \text{const.} \) so that (13) is satisfied. The nonzero components of the stress tensor are obtained from (11) as
\[
    \sigma_{\theta z} = -c \frac{\mu r z}{(r^2 + z^2)^{5/2}}, \quad \sigma_{r \theta} = -c \frac{\mu r^2}{(r^2 + z^2)^{5/2}}.
\]

The constant \( c \) in (15) follows (14) and reads
\[
    c = \frac{3M}{2\pi\mu(2 - 3\cos \alpha + \cos^2 \alpha)}.
\]

5.7 Bending by couples (pure bending)

In this section we consider bending by couples of a prismatic rod having arbitrary cross-section. Suppose that the rod is loaded by a system of forces and couples, acting at its bases, that is equivalent to the couple \( \mathbf{M} = M_1 \mathbf{e}_1 + M_2 \mathbf{e}_2 \) and that the lateral surface of the rod is stress free. The expression for \( \mathbf{M} \) shows that we assumed there is no torsional component \((M_3)\) of the outer load (see Fig. 14). The axis \( x_3 \) passes through the centroids of the cross-section and is in the plane \( \Pi \).

We assume that the components of the stress tensor at an arbitrary point of the cross-section are
\[
    \sigma_{33} = ax_1 + bx_2; \quad \sigma_{22} = \sigma_{11} = \sigma_{12} = \sigma_{13} = \sigma_{23} = 0,
\]
where \( a \) and \( b \) are constants. If we cut the rod at the cross-section \( x_3 \) (see Fig. 14) and consider the equilibrium of the part I (i.e., the part \([0, x_3]\)) we obtain (see Fig. 15)
\[
    M_1 = \int_A \int_A \sigma_{33}x_2 dA; \quad M_2 = \int_A \int_A \sigma_{33}x_1 dA; \quad \int_A \int_A \sigma_{33} dA = 0.
\]
By using (1) in (2) we have

\[ a J_{12} + b J_{11} = M_1; \quad a J_{22} + b J_{12} = M_2; \]

where

\[ a \int_A x_1 dA + b \int_A x_2 dA = 0, \]

\[ J_{11} = \int_A x_2^2 dA; \quad J_{22} = \int_A x_1^2 dA; \quad J_{12} = \int_A x_1 x_2 dA, \]

are axial and centrifugal moments of inertia of the cross-sectional area \( A \). The condition (3)_3 is always satisfied since \( \bar{x}_1 \) and \( \bar{x}_2 \) are axes passing through the centroid of the cross-section. From (3)_1,2 it follows that

\[ a = \frac{M_2 J_{11} - M_1 J_{12}}{J_{11} J_{22} - J_{12}^2}; \quad b = \frac{M_1 J_{22} - M_2 J_{12}}{J_{11} J_{22} - J_{12}^2}. \]

With (5), the stress distribution that satisfies boundary conditions at \( x_3 = 0 \) and \( x_3 = l \) is

\[ \sigma_{33} = \frac{M_2 J_{11} - M_1 J_{12}}{J_{11} J_{22} - J_{12}^2} x_1 + \frac{M_1 J_{22} - M_2 J_{12}}{J_{11} J_{22} - J_{12}^2} x_2. \]

The stress tensor with components (1) satisfies the equilibrium equations (4.1-1) and the compatibility conditions (4.1-20). Since the rod is prismatic, we have (see Fig. 2)

\[ \mathbf{n} = \cos \varphi \mathbf{e}_1 + \sin \varphi \mathbf{e}_2. \]
Therefore the stress vector at an arbitrary point of the lateral surface of the rod is

\[ \mathbf{p}_n = 0. \]  

(8)

We determine now the components of the displacement vector. To simplify the analysis we assume the axes \( \bar{x}_1, \bar{x}_2 \) are the principal axes of the cross-section so that \( J_{12} = 0 \) and that \( M_1 = 0, M_2 = M_s \). Then (6) becomes

\[ \sigma_{33} = \frac{M_s}{J_{22}} \bar{x}_1. \]  

(9)

With (9) and (3.4-14) we obtain

\[ \frac{\partial u_1}{\partial x_1} = -\nu \frac{M_s}{EJ_{22}} \bar{x}_1; \quad \frac{\partial u_2}{\partial x_2} = -\nu \frac{M_s}{EJ_{22}} \bar{x}_2; \quad \frac{\partial u_3}{\partial x_3} = \frac{M_s}{EJ_{22}} \bar{x}_1, \]  

(10)

and

\[ \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} = 0; \quad \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} = 0; \quad \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} = 0. \]  

(11)

By integration, we obtain from (10)

\[ u_3 = \frac{M_s}{EJ_{22}} x_1 x_3 + w_0(x_1, x_2), \]  

(12)

where \( w_0 \) is an arbitrary function. From (11) and (12) we have

\[ \frac{\partial u_1}{\partial x_3} = -\frac{M_s}{EJ_{22}} x_3 - \frac{\partial w_0}{\partial x_1} \bar{x}_3 + u_0(x_1, x_2); \quad \frac{\partial u_2}{\partial x_3} = -\frac{\partial w_0}{\partial x_2} \bar{x}_3 + u_0(x_1, x_2). \]  

(13)

By integrating (13) it follows that

\[ u_1 = -\frac{M_s}{EJ_{22}} x_3^2 - \frac{\partial w_0}{\partial x_1} \frac{x_3}{2} + u_0(x_1, x_2); \quad u_2 = -\frac{\partial w_0}{\partial x_2} x_3 + u_0(x_1, x_2). \]  

(14)

Substituting (14) into (10) we obtain

\[ \frac{\partial^2 w_0}{\partial x_1^2} x_3 + \frac{\partial w_0}{\partial x_1} = -\nu \frac{M_s}{EJ_{22}} \bar{x}_1; \quad \frac{\partial^2 w_0}{\partial x_2^2} x_3 + \frac{\partial w_0}{\partial x_2} = -\nu \frac{M_s}{EJ_{22}} \bar{x}_1. \]  

(15)

From (15) we conclude that

\[ \frac{\partial^2 w_0}{\partial x_1^2} = \frac{\partial^2 w_0}{\partial x_2^2} = 0; \quad \frac{\partial w_0}{\partial x_1} + \frac{\nu M_s}{EJ_{22}} \bar{x}_1 = 0; \quad \frac{\partial w_0}{\partial x_2} + \frac{\nu M_s}{EJ_{22}} \bar{x}_1 = 0. \]  

(16)
By integrating (16)$_{2,3}$ we get
\[
 u_0 = -\frac{\nu M_s}{EJ_{22}} \frac{x_1^2}{2} + \varphi_1(x_2);
\]
\[
 v_0 = -\frac{\nu M_s}{EJ_{22}} x_1 x_2 + \varphi_2(x_1). \tag{17}
\]

Substituting (17) into (14) we obtain
\[
 u_1 = -\frac{\nu M_s}{EJ_{22}} \frac{x_3^2}{2} - \frac{\partial w_0}{\partial x_1} x_3 - \frac{\nu M_s}{EJ_{22}} \frac{x_1^2}{2} + \varphi_1(x_2);
\]
\[
 u_2 = -\frac{\partial w_0}{\partial x_2} x_3 - \frac{\nu M_s}{EJ_{22}} x_1 x_2 + \varphi_2(x_1). \tag{18}
\]

Finally by using (18) in (11)$_1$ we get
\[
 -\frac{\partial^2 w_0}{\partial x_1 \partial x_2} x_3 + \frac{d\varphi_1}{dx_2} - \frac{\partial^2 w_0}{\partial x_1 \partial x_2} x_3 - \frac{\nu M_s}{EJ_{22}} x_2 + \frac{d\varphi_2}{dx_1} = 0. \tag{19}
\]

From (19) it follows that
\[
 \frac{\partial^2 w_0}{\partial x_1 \partial x_2} = 0; \quad \frac{d\varphi_1}{dx_2} + \frac{d\varphi_2}{dx_1} - \frac{\nu M_s}{EJ_{22}} x_2 = 0. \tag{20}
\]

By solving (20) and by substituting the result in (12) and (18) we obtain
\[
 u_1 = -\frac{M_s}{2EJ_{22}} \left(\nu x_1^2 + x_3^2 - \nu x_2^2\right) - c_2 x_3 + c_4 x_2 + c_6;
\]
\[
 u_2 = -\frac{\nu M_s}{EJ_{22}} x_1 x_2 - c_1 x_3 - c_4 x_1 + c_5;
\]
\[
 u_3 = \frac{M_s}{EJ_{22}} x_1 x_3 + c_1 x_2 + c_2 x_1 + c_3, \tag{21}
\]

where $c_i$, $i = 1, \ldots, 6$ are constants. If we assume that the point $x_1 = x_2 = x_3 = 0$ is fixed and the rod is built in at the end $x_3 = 0$ we obtain, as a boundary condition, the following expressions
\[
 u_i(x_1 = 0; \; x_2 = 0, \; x_3 = 0) = 0; \quad i = 1, 2, 3
\]
\[
 \frac{\partial u_2}{\partial x_3} = 0; \quad \frac{\partial u_1}{\partial x_3} + \frac{\partial u_1}{\partial x_1} = 0. \tag{22}
\]

By using (21) in (22) we conclude that $c_i = 0$, $i = 1, \ldots, 6$ so that
\[
 u_1 = -\frac{M_s}{2EJ_{22}} \left(\nu x_1^2 + x_3^2 - \nu x_2^2\right);
\]
\[
 u_2 = -\frac{\nu M_s}{EJ_{22}} x_1 x_2; \quad u_3 = \frac{M_s}{EJ_{22}} x_1 x_3. \tag{23}
\]
The rod axis in the undeformed state is defined as \( x_1 = x_2 = 0 \). In the deformed state it becomes a curve described by

\[
\begin{align*}
  u_1 &= -\frac{M_s}{2EJ_{22}}x_3^2; \\
  u_2 &= u_3 = 0.
\end{align*}
\]  

(24)

The curvature of this curve is determined from (24) by using the following expression

\[
\kappa = \frac{1}{R} = \frac{\frac{d^2u_1}{dx_3^2}}{1 + \left(\frac{du_1}{dx_3}\right)^2}^{3/2} = -\frac{M_s}{EJ_{22}} \frac{\left(\frac{M_s x_3}{EJ_{22}}\right)^2}{1 + \left(\frac{M_s x_3}{EJ_{22}}\right)^2}^{3/2}.
\]  

(25)

If we assume that \( M_s/EJ_{22} \ll 1 \) then (25) becomes

\[
\kappa = \frac{1}{R} = -\frac{M_s}{EJ_{22}}.
\]  

(26)

Equation (26) is known as the basic constitutive equation of the Bernoulli-Euler rod theory (see Atanackovic (1997)). It could be expressed as: the curvature of the rod axis is proportional to the bending moment to which the rod is subjected. The analysis presented here shows that the constitutive equation (26) has its basis in the theory of elasticity.

5.8 Bending of a rod by a terminal load

Consider a prismatic rod fixed at one end and free at the other end. Suppose that at the free end the rod is loaded by a concentrated force \( F \) (or a system of forces whose resultant force is equal to \( F \)) and that the body forces are equal to zero (see Fig. 16). Let the coordinate system \( \bar{x}_i \) be oriented so that \( \bar{x}_3 \) coincides with the axis of the rod connecting the centroids of the cross-sectional area.

Figure 16
For simplicity we assume that the cross-section is symmetric with respect to the \( x_2 \) axis. The action line of the force \( F \) is parallel to the \( x_1 \) axis but displaced for an amount \( e \) (the eccentricity). In a cross-section of the rod that is at the distance \( x_3 \) from the free end we have the bending moment given as

\[
M_2 = F(l - x_3).
\]

We assume that the stress component \( \sigma_{33} \) is given as in the case of pure bending (see (5.7-9) with \( J_{22} = J \))

\[
\sigma_{33} = -\frac{M_2}{J} x_1 = -\frac{F(l - x_3)}{J} x_1.
\]

Also we assume that

\[
\sigma_{11} = \sigma_{22} = \sigma_{12} = 0.
\]

The equilibrium equations then become

\[
\frac{\partial \sigma_{13}}{\partial x_3} = 0; \quad \frac{\partial \sigma_{23}}{\partial x_3} = 0; \quad \frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{F x_1}{J} = 0.
\]

From (4)_1,2 it follows that \( \sigma_{13} \) and \( \sigma_{23} \) are the same in all cross-sections. The compatibility conditions (4.1-20), by taking into account (3), reduce to

\[
\nabla^2 \sigma_{13} + \frac{F}{(1 + \nu)J} = 0; \quad \nabla^2 \sigma_{23} = 0,
\]

where we used \( \Theta = \sigma_{11} + \sigma_{22} + \sigma_{33} = \sigma_{33} \). Since the rod is prismatic and the lateral surface \( S_L \) is stress free we have the following boundary condition

\[
p_n = 0 \quad \text{on} \quad S_L,
\]

or

\[
p_n = \begin{bmatrix} 0 & 0 & \sigma_{13} \\ 0 & 0 & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ 0 \end{bmatrix} = 0. \quad \text{on} \quad S_L.
\]

From (7) we obtain

\[
\sigma_{13} n_1 + \sigma_{23} n_2 = 0, \quad \text{on} \quad S_L.
\]

Equation (4)_3 we can write as

\[
\frac{\partial}{\partial x_1} \left\{ \sigma_{13} + \frac{F}{2J} \left[ x_1^2 - f(x_2) \right] \right\} + \frac{\partial \sigma_{23}}{\partial x_2} = 0,
\]

\[\text{If we omit this assumption, the analysis changes slightly. For example, instead of (2) we would have an expression of the type (5.7-1).}\]
where \( f \) is an arbitrary function. To solve (9) we introduce the stress function \( B = B(x_1, x_2) \) by the following expressions

\[
\sigma_{13} = \frac{\partial B}{\partial x_2} - \frac{F}{2J} [x_1^2 - f(x_2)]; \quad \sigma_{23} = -\frac{\partial B}{\partial x_1}.
\] (10)

With (10) the condition (9) is identically satisfied and (5) reduces to

\[
\frac{\partial}{\partial x_2} \nabla^2 B = \frac{F}{2J} \left( \frac{2\nu}{1+\nu} - \frac{d^2 f}{dx_2^2} \right); \quad \frac{\partial}{\partial x_1} \nabla^2 B = 0.
\] (11)

From (11) it follows that \( \nabla^2 B \) does not depend on \( x_1 \). Then, by integrating (11) we obtain

\[
\nabla^2 B = \frac{F}{2J} \left( \frac{2\nu}{1+\nu} x_2 - \frac{df}{dx_2} \right) + c_1,
\] (12)

where \( c_1 \) is a constant. The constant \( c_1 \) has an interesting physical interpretation. Consider the \( \bar{w}_3 \) component of the rotation vector (see (2.4-15))

\[
\bar{w}_3 = -\omega_{12} = \omega_{21} = \frac{1}{2} \left( \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right).
\] (13)

The change of \( \bar{w}_3 \) along the rod axis is determined from

\[
\frac{\partial \bar{w}_3}{\partial x_3} = \frac{1}{2} \left( \frac{\partial^2 u_2}{\partial x_1 \partial x_3} - \frac{\partial^2 u_1}{\partial x_2 \partial x_3} \right)
= \frac{1}{2} \frac{\partial}{\partial x_1} \left( \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} \right) - \frac{1}{2} \frac{\partial}{\partial x_2} \left( \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} \right)
= \frac{\partial E_{32}}{\partial x_1} - \frac{\partial E_{31}}{\partial x_2}.
\] (14)

From Hooke’s law (3.3-43) and (14) we obtain

\[
\frac{\partial \bar{w}_3}{\partial x_3} = \frac{1}{2\mu} \left[ \frac{\partial \sigma_{32}}{\partial x_1} - \frac{\partial \sigma_{31}}{\partial x_2} \right].
\] (15)

By using (10) in (15) we have

\[
\frac{\partial \bar{w}_3}{\partial x_3} = \frac{1}{2\mu} \left[ -\nabla^2 B + \frac{F x_1^2}{2J} - \frac{F}{2J} \frac{df}{dx_2} \right].
\] (16)

Finally by substituting (12) equation (16) becomes

\[
\frac{\partial \bar{w}_3}{\partial x_3} = \frac{\nu F}{EJ} x_2 + \frac{c_1}{2\mu}.
\] (17)
On the rod axis we have $x_2 = 0$ so that (16) could be interpreted as: the constant $c_1$ in (12) is connected with the local rotation of the cross-section about the $\bar{x}_3$ axis calculated at the centroid of the cross-section. Thus by using the notation of Section 5.5 (especially (5.5-1)) we have

$$c_1 = -2\mu\tilde{\theta}. \quad (18)$$

Equation (12) could now be written as

$$\nabla^2 B = \frac{F}{2J} \left( \frac{2\nu}{1 + \nu} x_2 - \frac{df}{dx_2} \right) - 2\mu\tilde{\theta}. \quad (19)$$

The boundary conditions corresponding to (19) follow from (7) and (10) so that we have

$$\begin{align*}
\left\{ \frac{\partial B}{\partial x_2} - \frac{F}{2J} [x_1^2 - f(x_2)] \right\} n_1 + \left( -\frac{\partial B}{\partial x_1} \right) n_2 &= 0. \quad (20)
\end{align*}$$

By using the expressions

$$n_1 = \frac{dx_2}{dS}; \quad n_2 = -\frac{dx_1}{dS}, \quad (21)$$

where $S$ is the arc length of the contour $C$ of the cross-section, we obtain

$$\frac{dB}{dS} = \frac{F}{2J} [x_1^2 - f(x_2)] \frac{dx_2}{dS}, \quad \text{on } C. \quad (22)$$

As is seen, the bending problem is reduced to the solution of the boundary value problem (19), (22). The function $f(x_2)$ is arbitrary and should be chosen so that right-hand side in (22) is simplified; for example, it is equal to zero. In general the bending leads to torsion of the rod, since $c_1$ (or \( \tilde{\theta} \)) is not equal to zero.

We turn now to the problem of determining an important point on the cross-section called the center of flexure. When the force $F$ passes through this point and has an action line parallel to the principal axis of a cross-section, then we have $\tilde{\theta} = 0$ (or $c_1 = 0$ in (12)). We denote the coordinates of the center of flexure as $(x_{1cf}, x_{2cf}, l)$. The position of this point is determined from the condition that the moment of the forces resulting from shear stresses $\sigma_{13}$ and $\sigma_{23}$ for the center of flexure is equal to zero. Thus, the coordinates $(x_{1cf}, x_{2cf})$ must satisfy

$$\int_A \int [\sigma_{13}(x_2 - x_{2cf}) - \sigma_{23}(x_1 - x_{1cf})] dA = 0. \quad (23)$$

We can write as (24) as

$$M + x_{1cf}Q_2 - x_{2cf}Q_1 = 0, \quad (24)$$
where

\[
M = \int_A \int \left[ \sigma_{13} x_2 - \sigma_{23} x_1 \right] dA, \quad Q_1 = \int_A \int \sigma_{13} dA; \quad Q_2 = \int_A \int \sigma_{23} dA,
\]

are the bending moment and the components of the resultant force in an arbitrary cross-section.

If the rod is loaded by a single force having the action line parallel to a principal axis of the cross-section (as shown in Fig. 17 parallel to the axis of symmetry) then \( x_{1\text{cf}} = 0 \) and

\[
Q_2 = 0; \quad Q_1 = F. \tag{26}
\]

From (24) it follows that

\[
x_{2\text{cf}} = \frac{M}{F}. \tag{27}
\]

The moment \( M \) we determine from (25), and (10). Thus by substituting (10) into (25), we obtain

\[
M = -\int_A \int \left( x_1 \frac{\partial B}{\partial x_1} + x_2 \frac{\partial B}{\partial x_2} \right) dA + \frac{F}{2J} \int_A \int \left[ x_1^2 - f(x_2) \right] x_2 dA. \tag{28}
\]

By transforming the first term on the right-hand side of (28) and by using Gauss’ theorem, we conclude that (see (5.5-36))

\[
M = 2 \int_A \int \int B(x_1, x_2) dA + \frac{F}{2J} \int_A \int \left[ x_1^2 - f(x_2) \right] x_2 dA. \tag{29}
\]

With (29) and (27) the center of flexure is determined. Note that we must solve the bending problem (determine \( B \) and \( f \)) before we determine \( x_{2\text{cf}} \).

We now present solutions to some concrete bending problems.

i) Rod with elliptic cross-section

Consider a prismatic rod with elliptic cross-section loaded by a force \( F \) passing through the centroid of the cross-section (\( e = 0 \)) whose contour \( C \) is given by

\[
\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1. \tag{30}
\]
We assume that the function $f(x_2)$ is

$$f(x_2) = a^2 \left(1 - \frac{x_2^2}{b^2}\right).$$  \hspace{1cm} (31)

Then, from (22) it follows that

$$B = 0 \hspace{1cm} \text{on } C.$$  \hspace{1cm} (32)

By using (31) in (19), with $\bar{\theta} = 0$, we obtain

$$\nabla^2 B = \frac{F}{J} \left(\frac{\nu}{1+\nu} + \frac{a^2}{b^2}\right) x_2.$$  \hspace{1cm} (33)

We assume the solution of (33) in the form

$$B = m \frac{F a^2}{J} \left(\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} - 1\right) x_2,$$  \hspace{1cm} (34)

where $m$ is a constant. By substituting (34) in (33) it follows that

$$m = \frac{\nu b^2 + (1+\nu) a^2}{2(1+\nu)(b^2 + 3a^2)}.$$  \hspace{1cm} (35)

Therefore, the function $B$ is

$$B = \frac{\nu b^2 + (1+\nu) a^2}{2(1+\nu)(b^2 + 3a^2)} \frac{F a^2}{J} \left(\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} - 1\right) x_2.$$  \hspace{1cm} (36)

From (10) we determine the shear stresses as

$$\sigma_{13} = \frac{b^2 + 2(1+\nu) a^2}{2(1+\nu)(b^2 + 3a^2)} \frac{F}{J} \left(a^2 - x_1^2 - \frac{(1-2\nu) a^2 x_2^2}{2(1+\nu)(a^2 + b^2)}\right);$$

$$\sigma_{23} = -\frac{\nu b^2 + (1+\nu) a^2}{(1+\nu)(b^2 + 3a^2)} \frac{F}{J} x_1 x_2.$$  \hspace{1cm} (37)

Note that in (37) we use $J$ to denote the principal moment inertia for the elliptical cross-section that is given as $J = \pi a^3 b/4$.

In the special case of circular cross-section, we have $a = b = R$ and the stresses (37) become

$$\sigma_{13} = \frac{3+2\nu}{8(1+\nu)} \frac{F}{J} \left(R^2 - x_1^2 - \frac{(1-2\nu) x_2^2}{3+2\nu}\right);$$

$$\sigma_{23} = -\frac{1+2\nu}{4(1+\nu)} \frac{F}{J} x_1 x_2.$$  \hspace{1cm} (38)
Next we determine the components of the displacement vector corresponding to the circular cross-section. From Hooke’s law and (2), (3) and (38) we get the components of the strain tensor as

\[ E_{11} = \frac{\nu F}{EJ} (l - x_3)x_1; \quad E_{22} = \frac{\nu F}{EJ} (l - x_3)x_1; \]
\[ E_{33} = -\frac{F}{EJ} (l - x_3)x_1; \quad E_{12} = 0; \]
\[ E_{13} = \frac{3 + 2\nu}{8} \frac{F}{EJ} \left( R^2 - x_1^2 - \frac{1 - 2\nu}{3 + 2\nu} x_2^2 \right); \quad E_{23} = -\frac{1 + 2\nu}{4} \frac{F}{EJ} x_1 x_2. \]  

(39)

The boundary conditions corresponding to the rod having \((x_1 = 0, x_2 = 0, x_3 = 0)\) as a fixed point and with the rotation vector at this point equal to zero, are

\[ u_1 = u_2 = u_3 = 0; \quad w_1 = w_2 = w_3 = 0, \]  

(40)

at \(x_1 = x_2 = x_3 = 0\). From (39), (4.1-3), and (40) the following components of the displacement vector are obtained

\[ u_1 = \frac{F}{2EJ} \left[ \nu(x_1^2 - x_2^2)(l - x_3) + lx_3^2 - \frac{1}{3} x_3^2 \right]; \]
\[ u_2 = \frac{F}{EJ} \nu (l - x_3)x_1 x_2; \]
\[ u_3 = -\frac{F}{EJ} \left[ x_1 (lx_3 - \frac{1}{2} x_2^3) + \frac{1}{4} (x_1^3 + x_2^3) x_1 \right]. \]  

(41)

Note that for \(x_1 = x_2 = 0, x_3 = l\) the value of \(u_1\) becomes

\[ u_1(x_1 = 0, x_2 = 0, x_3 = l) = \frac{Fl^3}{3EJ}, \]  

(42)

which is the result of elementary bending theory.

ii) Rod with rectangular cross-section

Consider a rod fixed at one end and loaded at the other end with rectangular cross-section \(x_2 \in [-a, a]; x_3 \in [-b, b]\) shown in Fig. 18. We assume that the force’s action line is passing through the centroid so that \(e = 0\). The contour of the cross-section \(C\), in this case is given by (see Fig. 18)

\[ (x_1^2 - a^2)(x_2^2 - b^2) = 0, \quad \text{on} \ C. \]  

(43)

In this case, since the force is located at the centroid we conclude that \(\sigma_{13}\) is the even and \(\sigma_{23}\) the odd function of \(x_1\) and \(x_2\). On the basis of this and (10) we conclude that \(B\) must be the odd function of \(x_2\) and the even function of \(x_1\). Next we assume that the function \(f\) in (19), (20) is

\[ f(x_2) = a^2. \]  

(44)
Then (19) becomes (with $\bar{\theta} = 0$)

$$\nabla^2 B = \frac{F}{J}\frac{\nu}{1+\nu}x_2. \quad (45)$$

The boundary conditions corresponding to (45) follow from (22) and read

$$\frac{dB}{dS} = 0; \quad \text{on} \ x_2 = \pm b, \quad (46)$$

since $dx_2/dS = 0$ for $x_2 = \pm b$. Also

$$\frac{dB}{dS} = 0; \quad \text{on} \ x_1 = \pm a, \quad (47)$$

because $dB/dS = (F/2J)[x_1^2 - a^2] = 0$ for $x_1 = \pm a$. We take the function $B$ as

$$B = B_1(x_2) + B_2(x_1, x_2), \quad (48)$$

where $B_2$ is a harmonic function ($\nabla^2 B_2 = 0$). Then from (45) we have

$$\frac{d^2 B_1}{dx_2^2} = \frac{\nu F}{(1+\nu)J}x_2, \quad (49)$$

with the boundary conditions

$$B_1(x_2 = b) = B_1(x_2 = -b) = 0. \quad (50)$$

The solution of (49), (50) is

$$B_1 = \frac{\nu F}{6(1+\nu)J}(x_2^3 - b^2x_2). \quad (51)$$
For the function $B_2$ we have
\[ \nabla^2 B_2 = 0, \tag{52} \]
with boundary conditions
\[ B_2(x_1, x_2 = \pm b) = 0; \quad B_2(x_1 = \pm a, x_2) = -B_1(x_2). \tag{53} \]
By separating variables in (52) we obtain
\[ B_2 = A \operatorname{ch}(kx_1) \sin(kx_2), \tag{54} \]
where $A$ and $k$ are constants. Note that $B_2$ is even with respect to $x_1$ and odd with respect to $x_2$. From (53) it follows that
\[ \sin(kb) = 0 \Rightarrow k = n\pi/b; \quad n = 1, 2, 3, \ldots \tag{55} \]
Therefore the solution of (52) is
\[ B_2 = \sum_{n=1}^{\infty} A_n \operatorname{ch}\left(\frac{n\pi x_1}{b}\right) \sin\left(\frac{n\pi x_2}{b}\right), \tag{56} \]
where $A_n$ are constants to be determined. The condition (53) leads to
\[ \sum_{n=1}^{\infty} A_n \operatorname{ch}\left(\frac{n\pi a}{b}\right) \sin\left(\frac{n\pi x_2}{b}\right) = \frac{\nu F}{6(1+\nu)} (b^2 x_2 - x_2^3). \tag{57} \]
By multiplying (57) by $\sin(m\pi x_2/b)$ and integrating the result in the interval $[-b, b]$ we obtain
\[ A_n = -\frac{(-1)^n}{\operatorname{ch}(n\pi a/b)} \frac{4\nu F b^3}{n^3 \pi^3 (1+\nu)} \tag{58} \]
Finally from (48), (51), (56), and (58) the function $B$ becomes
\[ B = \frac{\nu F}{6(1+\nu)} J \left[ x_2^3 - b^2 x_2 - \frac{12b^3}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \frac{\operatorname{ch}(n\pi x_1/b)}{\operatorname{ch}(n\pi a/b)} \sin\left(\frac{n\pi x_2}{b}\right) \right]. \tag{59} \]
With (59) and (10) the shear stresses are
\[ \sigma_{13} = \frac{F}{2J} \left( a^2 - x_1^2 \right) \]
\[ + \frac{\nu F}{6(1+\nu)} J \left[ 3x_2^3 - b^2 - \frac{12b^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \frac{\operatorname{ch}(n\pi x_1/b)}{\operatorname{ch}(n\pi a/b)} \cos\left(\frac{n\pi x_2}{b}\right) \right]; \]
\[ \sigma_{23} = \frac{2\nu F b^2}{\pi^2 (1+\nu) J} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \frac{\operatorname{sh}(n\pi x_1/b)}{\operatorname{ch}(n\pi a/b)} \sin\left(\frac{n\pi x_2}{b}\right). \tag{60} \]
In (60) the axial moment of inertia $J$ is

$$J = \frac{(2a)^3(2b)}{12} = \frac{4}{3}a^3b. \quad (61)$$

In the elementary (strength of materials) approach the results for shear stresses are

$$\sigma_{13} = \frac{F}{2J}(a^2 - x_1^2) = \frac{3F}{8a^3b}(a^2 - x_1^2); \quad \sigma_{23} = 0. \quad (62)$$

Expression (62) can be obtained from (60) if we neglect the terms under the summation sign.

5.9 Elementary singular solutions

We now present solutions of the equilibrium equations for three-dimensional elastic bodies that are called by Boussinesq the elementary solutions of the first and second kind. Those solutions have importance in the study of contact stresses.

i) Elementary solutions of the first kind

We consider the solution of the equilibrium equations in the form of Papkovich-Neuber. Then in (4.5-17) we take, as a special case, the following

$$\phi_0 = 0; \quad \phi_1 = 0; \quad \phi_2 = 0; \quad \phi_3 = \frac{C}{r}; \quad A = -1; \quad B = 4(1 - \nu), \quad (1)$$

where $C$ is a constant and $r = [x_1^2 + x_2^2 + x_3^2]^{1/2}$. Equation (4.5-17) takes the form

$$u_i = 4(1 - \nu)\frac{C}{r}\delta_{3i} - \frac{\partial}{\partial x_i}\left[\frac{x_3 C}{r}\right]. \quad (2)$$

We prove first that the function $\phi_3$ is harmonic. Thus, we calculate

$$\frac{\partial}{\partial x_i}\left(\frac{1}{r}\right) = -\frac{x_i}{r^3}; \quad \frac{\partial}{\partial x_j}\left(\frac{1}{r^3}\right) = -\frac{3x_j}{r^5}; \quad \frac{\partial}{\partial x_i\partial x_j}\left(\frac{1}{r}\right) = -\frac{\delta_{ij}}{r^3} + \frac{3x_i x_j}{r^5},$$

so that

$$\nabla^2\left(\frac{1}{r}\right) = \frac{\partial^2}{\partial x_1^2}\left(\frac{1}{r}\right) + \frac{\partial^2}{\partial x_2^2}\left(\frac{1}{r}\right) + \frac{\partial^2}{\partial x_3^2}\left(\frac{1}{r}\right) = -\frac{3}{r^3} + 3x_i x_i \frac{1}{r^5} = 0. \quad (3)$$

By using (3) in (2) we obtain

$$u_1 = C\frac{x_1 x_3}{r^3}; \quad u_2 = C\frac{x_2 x_3}{r^3}; \quad u_3 = \frac{C}{r}\left[3 - 4\nu + \frac{x_3^2}{r^2}\right]. \quad (5)$$
With the components of the displacement vector (5) the strain tensor could be easily calculated and its components are

\[ E_{11} = \frac{C x_1^3}{r^3} \left( 1 - 3 \frac{x_1^2}{r^2} \right); \quad E_{22} = \frac{C x_2^3}{r^3} \left( 1 - 3 \frac{x_2^2}{r^2} \right); \quad E_{33} = \frac{C x_3^3}{r^3} \left( 4 \nu - 1 - 3 \frac{x_3^2}{r^2} \right); \]
\[ E_{12} = -3C \frac{x_1 x_2 x_3}{r^5}; \quad E_{13} = C \frac{x_1}{r^3} \left( 2 \nu - 1 - 3 \frac{x_3^2}{r^2} \right); \quad E_{23} = C \frac{x_2}{r^3} \left( 2 \nu - 1 - 3 \frac{x_3^2}{r^2} \right). \]

From (6) we obtain

\[ \vartheta = E_{11} + E_{22} + E_{33} = -2C \frac{x_3^3}{r^3} (1 - 2 \nu). \]  

By using (6),(7), Hooke’s law (3.3-44), and (see Problem 3 of Chapter 3) we obtain the components of the stress tensor in the form

\[ \sigma_{11} = 2\mu C \frac{x_1^3}{r^3} \left( 1 - 2 \nu - 3 \frac{x_1^2}{r^2} \right); \quad \sigma_{22} = 2\mu C \frac{x_2^3}{r^3} \left( 1 - 2 \nu - 3 \frac{x_2^2}{r^2} \right); \]
\[ \sigma_{33} = 2\mu C \frac{x_3^3}{r^3} \left( 2 \nu - 1 - 3 \frac{x_3^2}{r^2} \right); \quad \sigma_{12} = -6\mu C \frac{x_1 x_2 x_3}{r^5}; \]
\[ \sigma_{23} = 2\mu C \frac{x_2^3}{r^3} \left( 2 \nu - 1 - 3 \frac{x_2^2}{r^2} \right); \quad \sigma_{13} = 2\mu C \frac{x_1 x_3}{r^3} \left( 2 \nu - 1 - 3 \frac{x_3^2}{r^2} \right). \]  

As is seen from (5) and (8) the components of the displacement vector and components of the stress tensor increase to infinity at the origin of the coordinate system, that is, for \( r \to 0 \). Therefore the displacement and stress fields make sense only if we exclude the point \( r = 0 \). To make the solution regular we assume that there is a spherical cavity of small radius \( r_0 \) at the origin of the coordinate system. The unit normal on the surface of this cavity is

\[ n_i = - \frac{x_i}{r_0}. \]  

The stress vector is \( \mathbf{p}_n = \sigma \mathbf{n} \), so that its components are

\[ p_1 = 6C \mu \frac{x_1 x_3}{r_0^4}; \quad p_2 = 6C \mu \frac{x_2 x_3}{r_0^4}; \quad p_3 = 6C \mu \frac{x_3^3}{r_0^4} + 2(1 - 2 \nu) \frac{C \mu}{r_0^2}. \]  

From (10) we obtain the components of the resultant of surface forces acting on the spherical cavity as

\[ F_1 = \int_S \int_S p_1 dA = \frac{6C \mu}{r_0^4} \int_S \int_S x_1 x_3 dA; \quad F_2 = \int_S \int_S p_2 dA = \frac{6C \mu}{r_0^4} \int_S \int_S x_2 x_3 dA; \]
\[ F_3 = \int_S \int_S p_3 dA = \frac{6C \mu}{r_0^4} \int_S \int_S x_3^3 dA + 2(1 - 2 \nu) \frac{C \mu}{r_0^2} \int_S \int_S dA. \]
Because of symmetry we have

$$\int_s x_1 x_3 dA = 0; \quad \int_s x_2 x_3 dA = 0; \quad \int_s x_1^2 dA = \int_s x_2^2 dA = \int_s x_3^2 dA.$$  \hfill (12)

By using (12) it follows that

$$\int_s x_1^2 dA = \frac{1}{3} \int_s [x_1^2 + x_2^2 + x_3^2] dA = \frac{1}{3} \int_s r_0^2 dA = \frac{4\pi}{3} r_0^4.$$  \hfill (13)

Therefore the components of the resultant force are

$$F_1 = F_2 = 0; \quad F_3 = F = 16\pi C(1 - \nu).$$  \hfill (14)

From (14) we express $C$ in terms of $F$ as

$$C = \frac{F}{16\pi \mu(1 - \nu)}.$$  \hfill (15)

The components of the stress tensor (8) now become

$$\sigma_{11} = \frac{Fx_3}{8\pi(1 - \nu)r^3} \left(1 - 2\nu - 3\frac{x_1^2}{r^2}\right);$$

$$\sigma_{22} = \frac{Fx_3}{8\pi(1 - \nu)r^3} \left(1 - 2\nu - 3\frac{x_2^2}{r^2}\right);$$

$$\sigma_{33} = \frac{Fx_3}{8\pi(1 - \nu)r^3} \left(2\nu - 1 - 3\frac{x_3^2}{r^2}\right);$$

$$\sigma_{12} = -\frac{3F}{8\pi(1 - \nu)} \frac{x_1 x_2 x_3}{r^5};$$

$$\sigma_{23} = \frac{Fx_2}{8\pi(1 - \nu)r^3} \left(2\nu - 1 - 3\frac{x_3^2}{r^2}\right);$$

$$\sigma_{13} = \frac{Fx_1}{8\pi(1 - \nu)r^3} \left(2\nu - 1 - 3\frac{x_3^2}{r^2}\right).$$  \hfill (16)

The stress field (16) corresponds to an infinite elastic body loaded at the origin of the coordinate system by a concentrated force directed along the $x_3$ axis, that is, $F = Fe_3$. It is called the elementary solution of the first kind. It could be obtained from Kelvin’s particular solution of nonhomogeneous Lamé equations.

ii) Elementary solutions of the second kind

Consider a deformation field in the body when the solution is assumed in the Papkovich-Neuber form (4.5-17) with $A = 1$ and

$$\phi_0 = b \ln(r + x_3); \quad \phi_1 = 0; \quad \phi_2 = 0; \quad \phi_3 = 0,$$  \hfill (17)
where \( b \) is an arbitrary constant. We first prove that the function \( \phi_0 \) is harmonic. Thus, we calculate

\[
\frac{\partial \phi_0}{\partial x_1} = \frac{bx_1}{r(r + x_3)}; \quad \frac{\partial \phi_0}{\partial x_2} = \frac{bx_2}{r(r + x_3)}; \quad \frac{\partial \phi_0}{\partial x_3} = \frac{b}{r}.
\]

Also

\[
\frac{\partial^2 \phi_0}{\partial x_1^2} = \frac{b}{r^2(r + x_3)} - bx_1 \frac{2x_1 + x_1 x_3}{r^2(r + x_3)^2};
\]
\[
\frac{\partial^2 \phi_0}{\partial x_2^2} = \frac{b}{r^2(r + x_3)} - bx_2 \frac{2x_2 + x_2 x_3}{r^2(r + x_3)^2}; \quad \frac{\partial^2 \phi_0}{\partial x_3^2} = -\frac{b}{r^3}.
\]

From (19) it follows that \( \nabla^2 \phi_0 = 0 \). Now by using (17) and (4.5-17) we obtain

\[
u_1 = \frac{bx_1}{r(r + x_3)}; \quad \nu_2 = \frac{bx_2}{r(r + x_3)}; \quad \nu_3 = \frac{b}{r}.
\]

The strain tensor in the present case has components

\[
E_{11} = b \left[ \frac{x_1^2 + x_3^2}{r^3(r + x_3)} - \frac{x_1^2}{r^2(r + x_3)^2} \right]; \quad E_{22} = b \left[ \frac{x_2^2 + x_3^2}{r^3(r + x_3)} - \frac{x_2^2}{r^2(r + x_3)^2} \right];
\]
\[
E_{33} = -\frac{b x_3}{r^3}; \quad E_{12} = -\frac{b x_1 x_2 (2r + x_3)}{r^3(r + x_3)^2}; \quad E_{13} = -\frac{b x_1}{r^3}; \quad E_{23} = -\frac{b x_2}{r^3}.
\]

Note that

\[
\partial = E_{11} + E_{22} + E_{33} = \nabla^2 \phi_0 = 0,
\]

so that Hooke’s law gives

\[
\sigma_{11} = 2\mu b \left[ \frac{x_1^2 + x_3^2}{r^3(r + x_3)} - \frac{x_1^2}{r^2(r + x_3)^2} \right];
\]
\[
\sigma_{22} = 2\mu b \left[ \frac{x_2^2 + x_3^2}{r^3(r + x_3)} - \frac{x_2^2}{r^2(r + x_3)^2} \right];
\]
\[
\sigma_{33} = -2\mu b \frac{x_3}{r^3}; \quad \sigma_{12} = -2\mu b \frac{x_1 x_2 (2r + x_3)}{r^3(r + x_3)^2};
\]
\[
\sigma_{13} = -2\mu b \frac{x_1}{r^3}; \quad \sigma_{23} = -2\mu b \frac{x_2}{r^3}.
\]

The functions (20) satisfy the Lamé equations at all points of the elastic body except at the point \((0,0,0)\) and points on the negative \(x_3\) axis. Boussinesq called (20), (23) the elementary solution of the second kind. On the sphere of radius \(r_0\) the stress vector has components given by

\[
p_1 = 2\mu b \frac{x_1}{r_0^2(r_0 + x_3)}; \quad p_2 = 2\mu b \frac{x_2}{r_0^2(r_0 + x_3)}; \quad p_3 = 2\mu b \frac{1}{r_0^2}.
\]
5.10 The Boussinesq problem

Consider the half space \( x_3 > 0 \) occupied by an elastic medium. Suppose that a concentrated force \( \mathbf{F} \) is acting at the origin and that it is oriented in the positive \( x_3 \) direction (see Fig. 19). We want to determine stress and displacement fields. This constitutes the Boussinesq problem. Since the point 0 where the load is applied is a singular point (see Section 5.1) we remove a part of the space in the form of a hemisphere of radius \( r_0 \). Consider then a semi-infinite region bounded by the hemisphere and \( x_1 - x_2 \) plane.

![Figure 19](image)

We determine the stress and displacement fields by using the superposition of elementary solutions of the first and second kind. We assume that the force \( \mathbf{F} \) is a resultant of distributed stress vectors acting on the hemisphere. The stress vector \( \mathbf{p} \) acting on the hemisphere of radius \( r_0 \) is given by (5.9-10); that is,

\[
\begin{align*}
\bar{p}_1 &= 6C \mu \frac{x_1 x_3}{r_0^4}; \\
\bar{p}_2 &= 6C \mu \frac{x_2 x_3}{r_0^4}; \\
\bar{p}_3 &= 6C \mu \frac{x_3^2}{r_0^4} + 2(1 - 2\nu) \frac{C\mu}{r_0^2},
\end{align*}
\]

and the stress vector components on the plane \( x_3 = 0 \) follow from (5.9-8) and read

\[
\begin{align*}
\bar{\sigma}_{13} &= -2\mu C \frac{x_1}{r_0^2} (1 - 2\nu); \\
\bar{\sigma}_{23} &= -2\mu C \frac{x_2}{r_0^2} (1 - 2\nu); \\
\bar{\sigma}_{33} &= 0.
\end{align*}
\]

Suppose now that the stress vector \( \mathbf{\bar{p}} \) of the elementary solution of the second kind is also acting on the surface of the hemisphere so that (see (5.9-24))

\[
\begin{align*}
\bar{\bar{p}}_1 &= 2b \mu \frac{x_1}{r_0^2(r_0 + x_3)}; \\
\bar{\bar{p}}_2 &= 2b \mu \frac{x_2}{r_0^2(r_0 + x_3)}; \\
\bar{\bar{p}}_3 &= 2b \mu \frac{1}{r_0^2}.
\end{align*}
\]

The stress vector on the plane \( x_3 = 0 \) in this case becomes (see (5.9-23))

\[
\begin{align*}
\bar{\bar{\sigma}}_{13} &= -2b \mu \frac{x_1}{r_0^2}; \\
\bar{\bar{\sigma}}_{23} &= -2b \mu \frac{x_2}{r_0^2}; \\
\bar{\bar{\sigma}}_{33} &= 0.
\end{align*}
\]
If we require that the plane \( x_3 = 0 \) is stress free then

\[
\sigma_{13} = \bar{\sigma}_{13} + \bar{\sigma}_{13} = 0; \quad \sigma_{23} = \bar{\sigma}_{23} + \bar{\sigma}_{23} = 0; \quad \sigma_{33} = \bar{\sigma}_{33} + \bar{\sigma}_{33} = 0. \tag{5}
\]

By using (2) and (4) in (5) we conclude that

\[
b = -C(1 - 2\nu). \tag{6}
\]

We determine next the resultant force \( F \) of all stress vectors acting on the hemisphere. Because of symmetry we have

\[
F_3 = F = \int_{S/2} \int_S (\bar{p}_3 + \bar{p}_3) dA, \tag{7}
\]

where \( S/2 \) is the area of the hemisphere. Since

\[
\bar{p}_3 + \bar{p}_3 = 6C\mu \frac{x_3^2}{r_0^4}, \tag{8}
\]

we obtain from (7)

\[
F = \frac{6C\mu}{r_0^4} \int_{S/2} \int_S x_3^2 dA = \frac{1}{2} \frac{6C\mu}{r_0^4} \int_S x_3^2 dA = 4\pi C\mu, \tag{9}
\]

where we used (5.9-13). From (9) we determine \( C \) and from (6) the value of \( b \). Then by adding (5.9-5) and (5.9-20) we have

\[
u_1 = \frac{F}{4\pi \mu} \left[ \frac{x_1 x_3}{r^3} - (1 - 2\nu) \frac{x_1}{r(r + x_3)} \right];
\]

\[
u_2 = \frac{F}{4\pi \mu} \left[ \frac{x_2 x_3}{r^3} - (1 - 2\nu) \frac{x_2}{r(r + x_3)} \right];
\]

\[
u_3 = \frac{F}{4\pi \mu} \left[ \frac{2(1 - \nu)}{r} + \frac{x_3^2}{r^3} \right]. \tag{10}
\]

The components of the stress tensor for the displacement field (10) are

\[
\sigma_{11} = \frac{F}{2\pi} \left( \frac{x_3}{r^3} \left( 1 - 2\nu - 3 \frac{x_3^2}{r^2} \right) - (1 - 2\nu) \left[ \frac{x_1^2 + x_3^2}{r^3(r + x_3)} - \frac{x_1^2}{r^2(r + x_3)^2} \right] \right); \tag{11}
\]

\[
\sigma_{22} = \frac{F}{2\pi} \left( \frac{x_3}{r^3} \left( 1 - 2\nu - 3 \frac{x_2^2}{r^2} \right) - (1 - 2\nu) \left[ \frac{x_2^2 + x_3^2}{r^3(r + x_3)} - \frac{x_2^2}{r^2(r + x_3)^2} \right] \right); \tag{11}
\]

\[
\sigma_{33} = -3F \frac{x_3^3}{2\pi r^5}; \quad \sigma_{12} = \frac{F}{2\pi} \left[ (1 - 2\nu) \frac{x_1 x_2 (2r + x_3)}{r^3(r + x_3)^2} - 3 \frac{x_1 x_2 x_3}{r^5} \right]; \tag{11}
\]

\[
\sigma_{13} = -3F \frac{x_1 x_3^2}{2\pi r^5}; \quad \sigma_{23} = \frac{3F x_2 x_3^2}{2\pi r^5}. \tag{11}
\]
Note that in the solution (10), (11) the displacement field vanishes as $1/r$ and the stress field vanishes as $1/r^2$ when $r \to \infty$. On the plane $x_3 = 0$ the displacement vector has components

$$u_1 = -\frac{F}{4\pi\mu} \frac{(1 - 2\nu)x_1}{r^2}; \quad u_2 = -\frac{F}{4\pi\mu} \frac{(1 - 2\nu)x_2}{r^2}; \quad u_3 = \frac{F}{2\pi\mu} \frac{(1 - \nu)}{r}.$$  \hspace{1cm} (12)

We now analyze the properties of the solution of the Boussinesq problem. First, consider a point $K$ in the plane $x_3 = \text{const.} \neq 0$ with coordinates $(x_1, x_2, x_3)$. Let $R$ be the distance between the point on the $x_3$ axis with coordinates $(0,0,x_3)$ and $K$; that is,

$$R = \sqrt{x_1^2 + x_2^2}. \hspace{1cm} (13)$$

The intensity of the shear stress at $K$, for the plane $x_3 = \text{const.}$ is

$$\tau_3 = \sqrt{\sigma_{13}^2 + \sigma_{23}^2} = \frac{3F}{2\pi} \frac{Rx_3^2}{r^5}. \hspace{1cm} (14)$$

and has the direction of radius $R$. Second, note that from (10) it follows that the displacement for a point $K_1$ in the plane $x_3 = 0$ that is on the distance $r$ from the origin is

$$u = \sqrt{u_1^2 + u_2^2} = \frac{1 - 2\nu}{\mu} \frac{F}{4\pi r} = \frac{(1 + \nu)(1 - 2\nu)}{E} \frac{F}{2\pi r}.$$  \hspace{1cm} (15)

The displacement (15) has the direction of the vector connecting $O$ and $K_1$ and is directed toward $O$. Note also that (12)$_3$ could be written as

$$u_3 = \frac{F}{\pi E} \frac{(1 - \nu^2)}{r}. \hspace{1cm} (16)$$

Equation (16) is called the Boussinesq equation. It shows that in the plane $x_3 = 0$ we have

$$u_3 r = \text{const.} \hspace{1cm} (17)$$

We now analyze the generalization of the Boussinesq problem. Namely, consider the case of continuous distribution of normal loads over the part $\Omega$ of the plane $x_3 = 0$ (see Fig. 20). The force $dF$ acting on the part of the boundary $d\Omega$ is given by the following relation

$$dF = q(\xi, \eta) d\Omega, \hspace{1cm} (18)$$

where $q(\xi, \eta)$ is the specific (per unit area) load intensity.
The displacement vector in this case could be obtained by using the superposition of displacement fields corresponding to $dF$. Thus, we replace $x_1$, $x_2$, and $r$ by

$$\hat{x}_1 = x_1 - \xi; \quad \hat{x}_2 = x_2 - \eta; \quad \hat{r} = [(x_1 - \xi)^2 + (x_2 - \eta)^2 + x_3^2]^{1/2}, \quad (19)$$

respectively, and integrate over $\Omega$. The resulting displacement field is obtained as

$$u_1 = \frac{1}{4\pi\mu} \int \int_\Omega \left[ \frac{\hat{x}_1 x_3}{\hat{r}^3} - (1 - 2\nu) \frac{\hat{x}_1}{\hat{r}(\hat{r} + x_3)} \right] q(\xi, \eta)d\Omega;$$

$$u_2 = \frac{1}{4\pi\mu} \int \int_\Omega \left[ \frac{\hat{x}_2 x_3}{\hat{r}^3} - (1 - 2\nu) \frac{\hat{x}_2}{\hat{r}(\hat{r} + x_3)} \right] q(\xi, \eta)d\Omega;$$

$$u_3 = \frac{1}{4\pi\mu} \int \int_\Omega \left[ \frac{2(1 - \nu)}{\hat{r}} + \frac{x_3^2}{\hat{r}^3} \right] q(\xi, \eta)d\Omega, \quad (20)$$

where $x_1$, $x_2$ are coordinates of the point whose displacement vector has components $u_i$ and $\hat{x}_1$, $\hat{x}_2$ and $\hat{r}$ are given by (19). From (20)_3 we conclude that on the plane $x_3 = 0$ we have

$$u_3 = \frac{(1 - \nu)}{2\pi\mu} \int \int_\Omega \frac{q(\xi, \eta)d\Omega}{[(x_1 - \xi)^2 + (x_2 - \eta)^2]^{1/2}}. \quad (21)$$

We use the expression (21) later.

5.11 Tangential force on the elastic half space

Consider an elastic half space $x_3 > 0$ as shown in Fig. 21. At the bounding plane $x_3 = 0$ there is a single concentrated force $F$ having the action line laying in the bounding plane. The problem of determining stress and
displacement fields in this case is known as the problem of Cerruti. To obtain the solution of the problem we assume the displacement field to be a combination of the Galerkin vector (4.5-9) and the Lamé displacement potential (4.5-14). Thus, we assume that the Lamé potential (see (4.5-14)) is

$$\phi = A \frac{x_1}{r + x_3},$$

(1)

where $r = (x_1^2 + x_2^2 + x_3^3)^{1/2}$ and that the components of the Galerkin vector are

$$H_1 = Br; \quad H_2 = 0; \quad H_3 = C \ln[x_1(r + x_3)],$$

(2)

where $A$, $B$, and $C$ are constants.

![Figure 21](image)

The displacement field corresponding to (1), (2) is

$$2\mu u_1 = -\frac{\partial \phi}{\partial x_1} + 2(1 - \nu)\nabla^2 H_1 - \frac{\partial}{\partial x_1} \left[ \frac{\partial H_1}{\partial x_1} + \frac{\partial H_3}{\partial x_3} \right];$$

$$2\mu u_2 = -\frac{\partial \phi}{\partial x_2} + 2(1 - \nu)\nabla^2 H_2 - \frac{\partial}{\partial x_2} \left[ \frac{\partial H_1}{\partial x_1} + \frac{\partial H_3}{\partial x_3} \right];$$

$$2\mu u_3 = -\frac{\partial \phi}{\partial x_3} + 2(1 - \nu)\nabla^2 H_3 - \frac{\partial}{\partial x_3} \left[ \frac{\partial H_1}{\partial x_1} + \frac{\partial H_3}{\partial x_3} \right].$$

(3)

The components of the strain tensor can now be determined for both displacement fields (1) and (2). The stress tensor is then determined as the sum of stress tensors corresponding to (1) and (2). After lengthy calculations, it follows that (see Hahn (1985))

$$\sigma_{ij} = \bar{\sigma}_{ij} + \tilde{\sigma}_{ij},$$

(4)

where

$$\bar{\sigma}_{ij} = \frac{\partial \phi}{\partial x_i \partial x_j} + \left( \frac{\nu}{1 - \nu} \nabla^2 \phi \right) \delta_{ij}$$

(5)
is the part of the stress tensor corresponding to displacement potential (1) and

\[ \bar{\sigma}_{ij} = \nu \frac{\partial^2 (\text{div} \mathbf{H})}{\partial x_i \partial x_j} \delta_{ij} - \frac{\partial^2 (\text{div} \mathbf{H})}{\partial x_i \partial x_j} + (1 - \nu) \frac{\partial^2}{\partial x_i \partial x_j} \left( \frac{\partial H_i}{\partial x_j} + \frac{\partial H_j}{\partial x_i} \right), \]  

(6)

is part of the stress tensor corresponding to the displacement potential (2). For the problem shown in Fig. 21 the boundary conditions are

\[ \sigma_{23} = 0, \quad \sigma_{33} = 0 \quad \text{for} \quad x_3 = 0. \]  

(7)

From (7) it follows (see Hahn (1985)) that

\[ A = \frac{F(1 - 2\nu)}{2\pi}; \quad B = \frac{F}{4\pi(1 - \nu)}; \quad C = \frac{F(1 - 2\nu)}{4\pi(1 - \nu)} \]  

(8)

By using (8) in (3) and (4) we obtain the displacement and stress tensor components as \((r = (x_1^2 + x_2^2 + x_3^2)^{1/2})\)

\[ u_1 = \frac{F}{4\pi \mu} \left\{ \frac{1}{r} + \frac{x_1^2}{r^3} + (1 - 2\nu) \frac{x_1}{r(r + x_3)^2} \right\}; \]

\[ u_2 = \frac{F}{4\pi \mu} \left\{ \frac{x_1 x_2}{r^3} - (1 - 2\nu) \frac{x_1 x_2}{r(r + x_3)^2} \right\}; \]

\[ u_3 = \frac{F}{4\pi \mu} \left\{ \frac{x_1 x_3}{r^3} + (1 - 2\nu) \frac{x_1}{r(r + x_3)} \right\}, \]  

(9)

and

\[ \sigma_{11} = \frac{F x_1}{2\pi r^3} \left[ \frac{3 x_1^2}{r^2} + \frac{1 - 2\nu}{(r + x_3)^2} \left( r^2 - \frac{2 r x_1^2}{(r + x_3)} - x_1^2 \right) \right]; \]

\[ \sigma_{22} = \frac{F x_1}{2\pi r^3} \left[ \frac{3 x_2^2}{r^2} + \frac{1 - 2\nu}{(r + x_3)^2} \left( 3 r^2 - \frac{2 r x_1^2}{(r + x_3)} - x_1^2 \right) \right]; \]

\[ \sigma_{33} = -\frac{3 F x_1 x_3^2}{2\pi r^5}; \quad \sigma_{13} = -\frac{3 F x_1^2 x_3}{2\pi r^5}; \quad \sigma_{23} = -\frac{3 F x_1 x_2 x_3}{2\pi r^5}; \]

\[ \sigma_{12} = \frac{F x_2}{2\pi r^3} \left[ \frac{3 x_1^2}{r^2} - \frac{1 - 2\nu}{(r + x_3)^2} \left( r^2 - \frac{2 r x_1^2}{r + x_3} - x_1^2 \right) \right]; \]

\[ (\sigma_{11} + \sigma_{22} + \sigma_{33}) = -\frac{F}{\pi} \frac{2 x_1}{r^3}. \]  

(10)

Here, again, the solution (9),(10) is singular at \( x_1 = x_2 = x_3 = 0 \). We note that both the Boussinesq and Cerruti problems belong to the class of problems of determining an equilibrium configuration of an infinite elastic body bounded by the plane on which surface traction is prescribed (see Love (1944)). For problems when the displacement vector is prescribed on the bounding plane see Love (1944, p. 239).
5.12 Equilibrium of a circular cone

The method of separation of variables can be successfully used to obtain the solution of equilibrium equations for the case when the elastic body is a circular cone. Suppose that the cone is defined in the spherical coordinate system by \(0 < \rho < \infty; 0 \leq \varphi \leq \alpha; -\pi \leq \theta \leq \pi\). We assume that the components of the displacement are proportional to \(1/\rho\) and that \(u_\rho\) and \(u_\varphi\) are proportional to \(\cos n\theta\) and \(u_\theta\) is proportional to \(\sin n\theta\), with \(n\) being an integer. If \(u_\rho\), \(u_\varphi\), and \(u_\theta\) are assumed in the form

\[
\begin{align*}
  u_\rho &= \frac{\cos n\theta}{\rho} \left[ -\frac{\lambda + 2\mu}{\mu} \frac{\rho^2}{\cos n\theta} + C \tan^n \frac{\varphi}{2} + D \cot^n \frac{\varphi}{2} \right]; \\
  u_\varphi &= \frac{\cos n\theta}{\rho \sin \varphi} \left[ -\frac{\lambda + 3\mu}{2\mu} \sin \varphi \frac{d}{d \varphi} \left( \frac{\rho^2}{\cos n\theta} \right) \\ &\quad + \cos \varphi \left( C \tan^n \frac{\varphi}{2} + D \cot^n \frac{\varphi}{2} \right) + G \tan^n \frac{\varphi}{2} + H \cot^n \frac{\varphi}{2} \right]; \\
  u_\theta &= \frac{\sin n\theta}{\rho \sin \varphi} \left[ \frac{\lambda + 3\mu}{2\mu} \frac{\rho^2}{\cos n\theta} \\ &\quad - \cos \varphi \left( C \tan^n \frac{\varphi}{2} - D \cot^n \frac{\varphi}{2} \right) - G \tan^n \frac{\varphi}{2} + H \cot^n \frac{\varphi}{2} \right],
\end{align*}
\]

where

\[
\Delta = \frac{\cos n\theta}{\rho^2} \left[ A(n + \cos \varphi \tan^n \frac{\varphi}{2} + B(n - \cos \varphi \cot^n \frac{\varphi}{2}) \right],
\]

and \(A\), \(B\), \(C\), \(D\), \(G\), and \(H\) are constants. Then the equilibrium equations (4.3-11) with zero body forces, that is,

\[
\begin{align*}
  (\lambda + 2\mu) \rho \sin \varphi \frac{\partial}{\partial \rho} \frac{\partial}{\partial \rho} - 2\mu \left[ \frac{\partial}{\partial \varphi} (\rho \omega_\varphi \sin \varphi) - \frac{\partial \omega_\varphi}{\partial \theta} \right] &= 0; \\
  (\lambda + 2\mu) \frac{1}{\sin \varphi} \frac{\partial}{\partial \varphi} - 2\mu \left[ \frac{\partial}{\partial \rho} (\rho \omega_\rho) - \frac{\partial \omega_\rho}{\partial \varphi} \right] &= 0; \\
  (\lambda + 2\mu) \sin \varphi \frac{\partial}{\partial \varphi} - 2\mu \left[ \frac{\partial \omega_\rho}{\partial \theta} - \frac{\partial}{\partial \rho} (\rho \omega_\theta \sin \varphi) \right] &= 0,
\end{align*}
\]

are satisfied (see Love (1944, p. 201) and Parton and Perlin (1984, vol II, p.17)).

Note that for the case of axial symmetry the components of the displacement vector are independent of \(\theta\) and \(u_\theta = 0\), so that we obtain (see (4.3-13))

\[
\begin{align*}
  (\lambda + 2\mu) \frac{\partial}{\partial \rho} + 2\mu \frac{1}{\rho \sin \varphi} \frac{\partial}{\partial \varphi} (\rho \omega_\varphi \sin \varphi) &= 0; \\
  (\lambda + 2\mu) \frac{\partial}{\partial \varphi} - 2\mu \frac{1}{\sin \varphi} \frac{\partial}{\partial \rho} (\rho \omega_\theta \sin \varphi) &= 0,
\end{align*}
\]
where
\[
\vartheta = \frac{1}{\rho^2} \frac{\partial}{\partial \rho} (\rho^2 u_\rho) + \frac{1}{\rho \sin \varphi} \frac{\partial}{\partial \varphi} (u_\varphi \sin \varphi); \quad \omega_\theta = \frac{1}{2\rho} \left[ \frac{\partial u_\rho}{\partial \varphi} - \frac{\partial}{\partial \rho} (p u_\varphi) \right].
\] (5)

Equations (4) can be written as
\[
\frac{\partial \vartheta}{\partial \rho} + \frac{K}{\rho^2 \sin \varphi} \frac{\partial}{\partial \varphi} (\omega \sin \varphi) = 0; \quad \frac{\partial \vartheta}{\partial \varphi} - K \frac{\partial \omega}{\partial \rho} = 0,
\] (6)

where
\[
K = \frac{2\mu}{\lambda + 2\mu}; \quad \omega = \rho \omega_\theta.
\] (7)

The general solution of (6) can be expressed in terms of Legendre polynomials (for details, see Rekech (1977, p. 36)).

We present a few specific solutions of (3) and (4).

i) Let the constants in (1) and (2) be: \( n = 0, C = 0, D = 0, G = 0, H = 0, A = -B = -F_3/[8\pi(\lambda + 2\mu)] \). By using this we obtain
\[
\Delta = -\frac{F_3 \cos \varphi}{4\rho^2 \pi (\lambda + 2\mu)}; \quad u_\rho = \frac{F_3 \cos \varphi}{4\pi \mu \rho};
\]
\[
u_\varphi = -\frac{\lambda + 3\mu}{2(\lambda + 2\mu)} \frac{F_3 \sin \varphi}{4\pi \mu \rho}; \quad u_\theta = 0.
\] (8)

The stress field corresponding to (8) is
\[
\sigma_{\rho \rho} = -\frac{3\lambda + 4\mu}{\lambda + 2\mu} \frac{F_3 \cos \varphi}{4\pi \rho^2}; \quad \sigma_{\varphi \varphi} = \sigma_{\theta \theta} = \frac{\mu}{\lambda + 2\mu} \frac{F_3 \cos \varphi}{4\pi \rho^2};
\]
\[
\sigma_{\varphi \rho} = \frac{\mu}{\lambda + 2\mu} \frac{F_3 \sin \varphi}{4\pi \rho^2}; \quad \sigma_{\theta \varphi} = \sigma_{\theta \rho} = 0.
\] (9)

The stress field (9) corresponds to a concentrated force \( F_3 \) acting parallel to the \( \bar{Z}_3 \) direction in an unbounded elastic medium. Thus (8),(9) represents the solution of the so-called Kelvin problem (see Section 5.9).

ii) The case \( n = 1, A = B = -F_1/[8\pi(\lambda + 2\mu)], C = 0, D = 0, G = 0, H = 0 \) leads to
\[
\Delta = -\frac{F_1}{4\pi (\lambda + 2\mu)} \frac{\cos \theta \sin \varphi}{\rho^2}; \quad u_\rho = \frac{F_1 \sin \varphi \cos \theta}{4\pi \mu \rho};
\]
\[
u_\varphi = \frac{\lambda + 3\mu}{2(\lambda + 2\mu)} \frac{F_1 \cos \varphi \cos \theta}{4\pi \mu \rho}; \quad u_\theta = -\frac{\lambda + 3\mu}{2(\lambda + 2\mu)} \frac{F_1 \sin \theta}{4\pi \mu \rho}.
\] (10)

The stress field determined from Hooke’s law and (10) is
\[
\sigma_{\rho \rho} = -\frac{3\lambda + 4\mu}{\lambda + 2\mu} \frac{F_1 \sin \varphi \cos \theta}{4\pi \rho^2}; \quad \sigma_{\varphi \varphi} = \sigma_{\theta \theta} = \frac{\mu}{\lambda + 2\mu} \frac{F_1 \sin \varphi \cos \theta}{4\pi \rho^2};
\]
\[
\sigma_{\rho \varphi} = \frac{\mu}{\lambda + 2\mu} \frac{F_1 \cos \varphi \cos \theta}{4\pi \rho^2}; \quad \sigma_{\rho \theta} = \frac{\mu}{\lambda + 2\mu} \frac{F_1 \cos \varphi \cos \theta}{4\pi \rho^2}; \quad \sigma_{\theta \varphi} = 0.
\] (11)
The stress field (11) corresponds to a concentrated force of intensity \( F_1 \) acting at the origin of the coordinate system and with the action line directed along the \( x_1 \) axis.

iii) Consider the problem of finding the axially symmetric solution of equations (3), that is, the solution of the problem (4). Thus, if we assume a solution such that the displacement vector has components that are proportional to \((1/\rho)\) we obtain

\[
\begin{align*}
  u_\rho &= \frac{F_3}{\rho}; \\
  u_\varphi &= -\frac{F_3}{\rho} \frac{\sin \varphi}{1 + \cos \varphi}; \\
  u_\theta &= 0. 
\end{align*}
\]  

The stress field corresponding to (12) is

\[
\begin{align*}
  \sigma_{\rho\rho} &= -2\mu \frac{F_3}{\rho^2}; \\
  \sigma_{\varphi\varphi} &= 2\mu \frac{F_3}{\rho^2} \frac{\cos \varphi}{1 + \cos \varphi}; \\
  \sigma_{\theta\theta} &= 2\mu \frac{F_3}{\rho^2} \frac{1}{1 + \cos \varphi}; \\
  \sigma_{\rho\varphi} &= 2\mu \frac{F_3}{\rho^2} \frac{\sin \varphi}{1 + \cos \varphi}; \\
  \sigma_{\rho\theta} &= \sigma_{\theta\varphi} = 0. 
\end{align*}
\]  

iv) Consider now the displacement field

\[
\begin{align*}
  u_\rho &= 0; \\
  u_\varphi &= -\frac{F_4}{\rho} \frac{\cos \varphi}{1 + \cos \varphi}; \\
  u_\theta &= \frac{F_4}{\rho} \frac{\sin \theta}{1 + \cos \varphi}. 
\end{align*}
\]  

The stress field corresponding to (14) is

\[
\begin{align*}
  \sigma_{\rho\rho} &= 0; \\
  \sigma_{\varphi\varphi} &= -\sigma_{\theta\theta} = -2\mu \frac{F_4}{\rho^2} \frac{(1 - \cos \varphi) \cos \theta}{(1 + \cos \varphi) \sin \theta}; \\
  \sigma_{\varphi\theta} &= 2\mu \frac{F_4}{\rho^2} \frac{(1 - \cos \varphi) \sin \theta}{(1 + \cos \varphi) \sin \varphi}; \\
  \sigma_{\rho\theta} &= -2\mu \frac{F_4}{\rho^2} \frac{\sin \theta}{1 + \cos \varphi}; \\
  \sigma_{\rho\varphi} &= 2\mu \frac{F_4}{\rho^2} \frac{\cos \theta}{1 + \cos \varphi}. 
\end{align*}
\]  

v) Consider the displacement field obtained from (1) for: \( n = 1, A = 0, B = 0, C = 0, D = 0, G = -H \neq 0 \). In this case we have

\[
\begin{align*}
  u_\rho &= \frac{F_5}{\rho} \frac{\sin \varphi \cos \theta}{1 + \cos \varphi}; \\
  u_\varphi &= \frac{F_5}{\rho} \cos \theta; \\
  u_\theta &= -\frac{F_5}{\rho} \sin \theta. 
\end{align*}
\]  

The stress field corresponding to (16) is

\[
\begin{align*}
  \sigma_{\rho\rho} &= -\sigma_{\varphi\varphi} = -2\mu \frac{F_5}{\rho^2} \frac{\sin \varphi \cos \theta}{1 + \cos \varphi}; \\
  \sigma_{\theta\theta} &= 0; \\
  \sigma_{\varphi\varphi} &= -\mu \frac{F_5}{\rho^2} \frac{\sin \varphi \sin \theta}{1 + \cos \varphi}; \\
  \sigma_{\rho\theta} &= \mu \frac{F_5}{\rho^2} \left( 2 - \frac{1}{1 + \cos \varphi} \right) \sin \theta; \\
  \sigma_{\rho\varphi} &= -\mu \frac{F_5}{\rho^2} \left( 2 - \frac{1}{1 + \cos \varphi} \right) \cos \theta. 
\end{align*}
\]
The solutions just presented may be combined to obtain new ones. For example, consider a cone loaded at its vertex as shown in Fig. 22. Suppose that the surface of the cone is defined as \( \varphi = \alpha \). Then by combining ii), iv), and v) we write the displacement field as

\[
\begin{align*}
    u_\rho &= \frac{F_1}{4\pi \mu} \sin \varphi \cos \theta + \frac{F_5}{\rho} \frac{\sin \varphi \cos \theta}{1 + \cos \varphi}; \\
    u_\varphi &= \frac{\lambda + 3\mu}{2(\lambda + 2\mu)} \frac{F_1}{4\pi \mu} \cos \varphi \cos \theta - \frac{F_4}{\rho} \frac{\sin \varphi}{1 + \cos \varphi} + \frac{F_5}{\rho} \cos \theta; \\
    u_\theta &= -\frac{\lambda + 3\mu}{2(\lambda + 2\mu)} \frac{F_1}{4\pi \mu} \sin \theta + \frac{F_4}{\rho} \frac{\sin \varphi}{1 + \cos \varphi} - \frac{F_5}{\rho} \sin \theta. 
\end{align*}
\] (18)

The boundary conditions on the surface of the cone, corresponding to the stress-free surface, read

\[
\sigma_{\varphi\varphi} = \sigma_{\varphi\rho} = \sigma_{\varphi\theta} = 0. 
\] (19)

![Figure 22](image)

By using the expressions (11), (15), and (17) in (19) we obtain a homogeneous system of linear algebraic equations for \( F_1, F_4, \) and \( F_5 \). From this system it follows that

\[
2F_4 \frac{1 - \cos \alpha}{\sin \alpha} - F_5 \sin \alpha = 0; \\
2F_4 - F_5 (1 + 2 \cos \alpha) - \frac{F_1}{4\pi (\lambda + 2\mu)} (1 + \cos \alpha) \cos \alpha = 0; \\
-2F_4 \frac{1 - \cos \alpha}{\sin \alpha} + 2F_5 \sin \alpha + \frac{F_1}{4\pi (\lambda + 2\mu)} (1 + \cos \alpha) \sin \alpha = 0. 
\] (20)

Solving (20) we have

\[
F_4 = -\frac{(1 + \cos \alpha)^2}{8\pi (\lambda + 2\mu)} F_1; \quad F_5 = -\frac{(1 + \cos \alpha)}{4\pi (\lambda + 2\mu)} F_1. 
\] (21)

The constant \( F_1 \) must be connected with the force \( F = Fe_1 \). Determining the resultant of the forces acting on the part of the cone with the center...
at the vertex of the cone and contained within the cone we conclude that this resultant must be equal to $F$. Thus, it follows that

$$F_1 = \frac{4F(\lambda + 2\mu)}{[(2 + \cos \alpha)\lambda + 2\mu(1 - \cos \alpha)]^2}. \quad (22)$$

The relations (18), (21), (22) determine the displacement field for the problem shown in Fig. (22). We note that for the case when $\alpha = \pi/2$ the results reduce those of Section 5.11.

5.13 Thermal stresses in a sphere and in a cylinder

In this section we present solutions of two problems for the case when the heat and motion equations are uncoupled. This theory is known as the theory of thermal stresses.

Suppose a hollow sphere of outer radius $R_o$ and inner radius $R_i$ has symmetric temperature field $\dot{\theta} = \dot{\theta}(\rho)$ that is time independent. Suppose further the inner and the outer surfaces of the sphere are stress free and that outer surface of the sphere is held at constant temperature $T_o$ and the inner surface is held at constant temperature $T_i$. We want to determine the stress field in the sphere. We start from equations (4.3-19) and (4.3-12); that is,

$$\begin{align*}
(\lambda + 2\mu)\rho \sin \varphi \frac{\partial \dot{\theta}}{\partial \rho} - 2\mu \left[ \frac{\partial \omega_\varphi}{\partial \theta} - \frac{\partial}{\partial \varphi} (\omega_\varphi \sin \varphi) \right] \\
+ \rho \sin \varphi \left[ \left( f_\rho - \rho_0 \frac{\partial^2 u_\rho}{\partial t^2} \right) - \gamma \frac{\partial \dot{\theta}}{\partial \rho} \right] = 0,
\end{align*} \quad (1)$$

and

$$\begin{align*}
\dot{\theta} = E_{\rho\rho} + E_{\varphi\varphi} + E_{\theta\theta} = \frac{1}{\rho^2} \frac{\partial}{\partial \rho} (\rho^2 u_\rho) + \frac{1}{\rho \sin \varphi} \left[ \frac{\partial}{\partial \varphi} (u_\varphi \sin \varphi) + \frac{\partial u_\theta}{\partial \theta} \right]. \quad (2)
\end{align*}$$

By assuming the radial symmetry of deformation $u_\rho \neq 0; u_\theta = 0; u_\varphi = 0$ and that all variables depend on $\rho$ only (all variables are also independent of time) we obtain

$$\frac{d}{d\rho} \left[ \frac{1}{\rho^2} \frac{d}{d\rho} (\rho^2 u) \right] = \frac{\gamma}{\lambda + 2\mu} \frac{dT}{d\rho}, \quad (3)$$

where we used $\ddot{\theta} = T$ and $u_\rho = u$. Note that the coefficient on the right-hand side of (3) could be written as $\gamma/(\lambda + 2\mu) = \alpha_t(1 + v)/(1 - v)$, where $\alpha_t$ is the coefficient of linear expansion (see (3.5-13)). By integrating (3) we obtain

$$u = \frac{\gamma}{\lambda + 2\mu} \frac{1}{\rho^2} \int_{R_i}^{R_o} T(m) m^2 dm + C_1 \rho + \frac{C_2}{\rho}, \quad R_i < \rho < R_o, \quad (4)$$
where $C_1$ and $C_2$ are arbitrary constants that will be determined from the boundary conditions. To determine $C_1$ and $C_2$ we write the stress strain relation (3.5-9) in the cylindrical coordinate system (see (3.6-2)) and comment after it

\[
\begin{align*}
\vartheta &= 2\frac{u}{\rho} + \frac{du}{d\rho}; \quad \sigma_{\rho\rho} = (\lambda\vartheta - \gamma T) + 2\mu\frac{du}{d\rho}; \\
\sigma_{\varphi\varphi} &= (\lambda\vartheta - \gamma T) + 2\mu\frac{u}{\rho}; \quad \sigma_{\theta\theta} = (\lambda\vartheta - \gamma T) + 2\mu\frac{u}{\rho}; \\
\sigma_{\rho\varphi} &= \sigma_{\varphi\theta} = \sigma_{\theta\rho} = 0. \quad (5)
\end{align*}
\]

By using (4) in (5) we obtain

\[
\begin{align*}
\sigma_{\rho\rho} &= -\frac{4\mu\gamma}{\lambda + 2\mu\rho^2} \int_{R_i}^{\rho} T(m)m^2 \, dm + C_1(3\lambda + 2\mu) - \frac{4\mu}{\rho^3} C_2; \\
\sigma_{\varphi\varphi} &= \frac{2\mu\gamma}{\lambda + 2\mu\rho^2} \int_{R_i}^{\rho} T(m)m^2 \, dm + C_1(3\lambda + 2\mu) \\
& \quad + C_2\frac{2\mu}{\rho^3} - \gamma \left(\frac{\lambda}{\lambda + 2\mu} - 1\right) T(\rho). \quad (6)
\end{align*}
\]

From the condition that the inner and outer surfaces are stress free we have

\[
\sigma_{\rho\rho}(R_i) = 0; \quad \sigma_{\rho\rho}(R_o) = 0. \quad (7)
\]

Substituting (6) into (7) it follows that

\[
\begin{align*}
C_1 &= \frac{4\mu\gamma R_i^3}{(3\lambda + 2\mu)(\lambda + 2\mu)(R_o^3 - R_i^3)} \int_{R_i}^{R_o} T(m)m^2 \, dm; \quad C_2 = \frac{\gamma R_o^3}{(\lambda + 2\mu)(R_o^3 - R_i^3)} \int_{R_i}^{R_o} T(m)m^2 \, dm. \quad (8)
\end{align*}
\]

The stress field in the sphere is determined from (6) and (8). In both of these equations the temperature field that follows from the steady solution of the heat conduction equation should be used. Thus, we have to determine $T$ from

\[
\nabla^2 T = 0, \quad (9)
\]

with $T = T_i$ and $T = T_o$ on the inner and outer surface of the sphere. With the assumption that $T = T(\rho)$ equation (8) leads to

\[
\frac{d}{d\rho} \left[ \rho^2 \frac{dT(\rho)}{d\rho} \right] = 0 \quad R_i < \rho < R_o, \quad (10)
\]

and

\[
T(R_i) = T_i; \quad T(R_o) = T_o. \quad (11)
\]

From (10), (11) we obtain the temperature field as

\[
T(\rho) = \frac{1}{1 - R_i/R_o} \left[ R_i T_i \left( \frac{R_o}{\rho} - 1 \right) + T_o \left( 1 - \frac{R_i}{\rho} \right) \right]. \quad (12)
\]
The temperature given by (12) has to be used in (8).

We consider next a cylinder with inner radius \( R_i \) and the outer radius \( R_o \). We assume that the cylinder is long so that the temperature field does not depend on the axial coordinate \( z \). We assume further that the temperature field is axially symmetric so that \( T = T(r) \). Equations (4.3-2), (4.3-18) for the case of plane state of strain (see Section 2.8) become

\[
(\lambda + 2\mu)r \frac{d\vartheta}{dr} = r\gamma \frac{dT}{dr},
\]

where

\[
\vartheta = E_{rr} + E_{\vartheta\vartheta} + E_{zz} = \frac{1}{r} \frac{d}{dr}(ru_r) = \frac{1}{r} \frac{d}{dr}(ru).
\]

From (13), (14) we obtain

\[
u = \frac{\gamma - 1}{\lambda + 2\mu} \frac{1}{r} \int_{R_i}^{r} T(m)mdm + C_1 r + \frac{C_2}{r},
\]

where \( C_1 \) and \( C_2 \) are constants. By using Hooke’s law in the cylindrical coordinate system (3.6-1) we have

\[
\vartheta = E_{rr} + E_{\vartheta\vartheta} + E_{zz} = \frac{u_r}{r} + \frac{1}{r} \frac{d}{d\vartheta} \vartheta + \frac{1}{r} \frac{d}{dz} z = u + \frac{du}{dr};
\]

\[
\sigma_{rr} = \lambda \vartheta - \gamma T + 2\mu \frac{dru_r}{dr}; \quad \sigma_{\vartheta\vartheta} = \lambda \vartheta - \gamma T + 2\mu \frac{ru_r}{r};
\]

\[
\sigma_{zz} = \lambda \vartheta - \gamma T.
\]

From (15) and (16) we obtain

\[
\sigma_{rr} = -\frac{2\mu\gamma}{\lambda + 2\mu} \frac{1}{r} \int_{R_i}^{r} T(m)mdm + 2C_1(\lambda + \mu) - 2\mu \frac{C_2}{r^2};
\]

\[
\sigma_{\vartheta\vartheta} = \frac{2\mu\gamma}{\lambda + 2\mu} \frac{1}{r} \int_{R_i}^{r} T(m)mdm - \frac{2\mu\gamma}{\lambda + 2\mu} T(r) + 2C_1(\lambda + \mu) + 2\mu \frac{C_2}{r^2};
\]

\[
\sigma_{zz} = -\frac{2\mu\gamma}{\lambda + 2\mu} T(r) + 2\lambda C_1.
\]

In order to have plane state of strain (\( u_z = 0 \)) the stress distribution \( \sigma_{zz} \) at the ends of the cylinder must satisfy a certain condition. Since \( \sigma_{zz} \) depends on \( r \) this condition could be satisfied in the mean. That is, we have to apply an axial force of intensity

\[
F_a = 2\pi \int_{R_i}^{R_o} \sigma_{zz}(r)rdr.
\]

Then, according to the Saint-Venant principle we will have the stress state in the cylinder described by (17)\textsubscript{1,2}. The temperature field in (17) is determined from the steady state heat conduction equation that in the cylindrical coordinate system with the assumptions stated (radial symmetry)
5.14 Plane harmonic waves in an elastic and thermoelastic body

We note that expressions (6) and (17) are sometimes used in the cases when the temperature fields do not satisfy the stationary heat conduction equations exactly but only in some approximate sense. For example, if the temperature changes “slowly,” equations (6) and (17) may be used to estimate the stresses.

5.14 Plane harmonic waves in an elastic and thermoelastic body

In this section we consider a dynamic problem of linear elasticity and thermoelasticity. We consider the plane waves propagating in the direction of the \( x_1 \) axis. We say that the motion of a body is a plane wave propagating in the \( x_1 \) direction if at every point of an arbitrary plane perpendicular to the \( x_1 \) axis the displacement vector components and the temperature (in the case of thermoelasticity) are the same. Thus the displacement field and the temperature depend on the distance of a point from the fixed plane and time.

Consider first the problem of wave propagation in an elastic material. If the region occupied by an elastic body is so large that the effects of boundaries can be neglected it is possible to represent a disturbance as a sum of two waves. To show this, consider equations of motion (4.1-6) for the case when there are no body forces; read

\[
\lambda + \mu \frac{\partial^2 \vartheta}{\partial x_j} + \mu \nabla^2 u_j - \rho_0 \frac{\partial^2 u_j}{\partial t^2} = 0. \tag{1}
\]

Suppose that \( \mathbf{u} \) is represented as a sum of two vector fields, one solenoidal \( \mathbf{u}_1 (\text{div} \mathbf{u}_1 = 0) \) and the other irrotational \( \mathbf{u}_2 (\text{curl} \mathbf{u}_2 = \nabla \times \mathbf{u}_2 = 0) \). We used this decomposition earlier (see (4.5-1)). Thus we consider

\[
\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2 = \nabla \times \Psi + \nabla \varphi. \tag{2}
\]
Since $\text{div} \, u_1 = \vartheta = 0$, we obtain from (1)

$$\mu \nabla^2 u_{1i} = \rho_0 \frac{\partial^2 u_{1i}}{\partial t^2}. \quad (3)$$

Equation (3) is the wave equation. The velocity of wave propagation is given by

$$c_1 = \sqrt{\frac{\mu}{\rho_0}}. \quad (4)$$

The waves propagating with velocity (4) are the equivoluminal waves (since $e_v = \vartheta = 0$; see (2.9-15)). Also, if we use $u_2$ in (1) we obtain

$$(\lambda + \mu) \nabla^2 u_{2i} = \rho_0 \frac{\partial^2 u_{2i}}{\partial t^2}. \quad (5)$$

Equation (5) is again the wave equation and the corresponding wave velocity is

$$c_2 = \sqrt{\frac{\lambda + 2\mu}{\rho_0}}. \quad (6)$$

The waves described by (5) are called the irrotational waves. Thus, in an infinite elastic medium the disturbances travel with the speeds given by (4) and (6). The waves described by (3) and (5) are not necessarily plane.

We turn now to the problem of plane waves. When the plane waves are propagating in the direction of the $x_1$ axis the components of the displacement vector must be of the form

$$u_i = F_i(x_1 - ct), \quad (7)$$

where $F_i$ are arbitrary functions and $c$ is the speed of the plane wave. By substituting (7) into (1) we obtain

$$(\rho_0 c^2 - \mu)(\rho_0 c^2 - \lambda - 2\mu) = 0. \quad (8)$$

From (8) it follows that: the plane waves can propagate with the velocity of either equivoluminal or irrotational waves.

Consider now the problem of thermoelastic waves. Thus, we are looking for solutions of (4.1-24) that with the zero body forces ($f = 0$) and the rate of heat generated in a unit volume equal to zero ($q = 0$) take the form

$$(\lambda + \mu) \frac{\partial \vartheta}{\partial x_j} + \mu \nabla^2 u_j - \gamma \frac{\partial T}{\partial x_j} = \rho_0 \frac{\partial^2 u_j}{\partial t^2};$$

$$\frac{\partial^2 T}{\partial x_1^2} + \frac{\partial^2 T}{\partial x_2^2} + \frac{\partial^2 T}{\partial x_3^2} - \eta \frac{\partial \vartheta}{\partial t} = \frac{1}{\kappa} \frac{\partial T}{\partial t}, \quad (9)$$

where $\vartheta = \partial u_1/\partial x_1 + \partial u_2/\partial x_2 + \partial u_3/\partial x_3$. The plane thermoelastic wave propagating in the $x_1$ direction is a motion such that the components of
the displacement vector and the temperature difference depend on $x_1$ and the time $t$; that is, $u_i = u_i(x_1, t)$, $T = T(x_1, t)$. With this assumption (9) becomes

$$
(\lambda + 2\mu)\frac{\partial^2 u_1}{\partial x_1^2} - \gamma \frac{\partial T}{\partial x_1} = \rho_0 \frac{\partial^2 u_1}{\partial t^2};
$$

$$
\mu \frac{\partial^2 u_2}{\partial x_1^2} = \rho_0 \frac{\partial^2 u_2}{\partial t^2};
$$

$$
\mu \frac{\partial^2 u_3}{\partial x_1^2} = \rho_0 \frac{\partial^2 u_3}{\partial t^2};
$$

$$
\frac{\partial^2 T}{\partial x_1^2} - \eta \frac{\partial^2 u_1}{\partial x_1 \partial t} = \frac{1}{\kappa} \frac{\partial T}{\partial t}.
$$

(10)

Consider now the transverse waves ($u_1 = 0$)

$$
u_1 = 0; \quad u_2 = \Re[u_2^*(x_1, \omega)e^{i\omega t}];
$$

$$
u_3 = \Re[u_3^*(x_1, \omega)e^{i\omega t}]; \quad T = \Re[T^*(x_1, \omega)e^{i\omega t}],
$$

(11)

where $\omega$ is the given frequency, $\Re(\cdot)$ denotes the real part of (\cdot) and $i = \sqrt{-1}$. It is easy to see that for the displacement field (11) we have

$$
\vartheta = 0,
$$

(12)

so that the transverse waves are equivoluminal. By substituting (12) into (10) we obtain

$$
\mu \frac{\partial^2 u_2}{\partial x_1^2} = \rho_0 \frac{\partial^2 u_2}{\partial t^2}; \quad \mu \frac{\partial^2 u_3}{\partial x_1^2} = \rho_0 \frac{\partial^2 u_3}{\partial t^2}; \quad \frac{\partial^2 T}{\partial x_1^2} = \frac{1}{\kappa} \frac{\partial T}{\partial t}.
$$

(13)

Thus, the motion and heat conduction equations are separated. By substituting (11) into (13) we obtain the following solutions

$$
u_2 = B_+ \exp \left[-i \omega \left(t - \frac{x_1}{c_1}\right)\right] + B_- \exp \left[-i \omega \left(t + \frac{x_1}{c_1}\right)\right];
$$

$$
u_3 = C_+ \exp \left[-i \omega \left(t - \frac{x_1}{c_1}\right)\right] + C_- \exp \left[-i \omega \left(t + \frac{x_1}{c_1}\right)\right];
$$

$$
T = T_+^0 \exp \left[-x_1 \sqrt{\frac{\omega}{2\kappa}} - i \omega \left(t - \frac{x_1}{\sqrt{2\kappa}\omega}\right)\right] + T_-^0 \exp \left[-x_1 \sqrt{\frac{\omega}{2\kappa}} - i \omega \left(t + \frac{x_1}{\sqrt{2\kappa}\omega}\right)\right],
$$

(14)

where $c_1 = \sqrt{\mu/\rho_0}$ and $B_+, B_-, C_+, C_-, T_+^0$, and $T_-^0$ are amplitudes of the waves propagating in the $+x_1$ and $-x_1$ directions with the frequency $\omega$. From (12),(13) we conclude that: the transverse waves described by (14)$_{1,2}$ in a thermoelastic body do not produce temperature changes. The waves are neither dispersed nor damped. The solution of the heat conduction equation from (14)$_3$ shows that the thermal waves are damped and the amplitude decreases with the distance.
Next we consider the longitudinal wave ($u_2 = u_3 = 0$),

$$
u_1 = \Re[u_1^*(x_1, \omega)e^{i\omega t}]; \quad u_2 = 0; \quad u_3 = 0; \quad T = \Re[T^*(x_1, \omega)e^{i\omega t}] . \quad (15)$$

From (15) we conclude that \( \vartheta \neq 0, \nabla \times \mathbf{u} = 0 \). By substituting (15) into (10) we obtain

$$
(\lambda + 2\mu) \frac{\partial^2 u_1^*}{\partial x_1^2} - \gamma \frac{\partial T^*}{\partial x_1} = -\rho_0 \omega^2 u_1^* ;
$$

$$
\frac{\partial^2 T^*}{\partial x_1^2} - i\eta \omega \frac{\partial^2 u_1^*}{\partial x_1} = -i\omega \frac{1}{\kappa} T^* . \quad (16)
$$

Let

$$
U_1^* = u^0 \exp[i k x_1]; \quad T^* = T^0 \exp[i k x_1], \quad (17)
$$

where \( u^0 \) and \( T^0 \) are constants. By substituting (17) into (16) we obtain two relations

$$
\frac{u^0}{T^0} = \frac{m k}{s^2 - k^2}; \quad \frac{u^0}{T^0} = -\frac{\eta k q i}{q - k^2} , \quad (18)
$$

where

$$
s^2 = \frac{\rho_0 \omega^2}{(\lambda + 2\mu)}; \quad m = \frac{\gamma}{\lambda + 2\mu}; \quad q = \frac{i \omega}{k} . \quad (19)
$$

From (18) we obtain the condition that \( k \) must satisfy

$$
k^4 - k^2 [s^2 + q(1 + r)] + q s^2 = 0 , \quad (20)
$$

where

$$
\eta = \eta m k . \quad (21)
$$

The roots of equation (20) are

$$
k_{1,2}^2 = s^2 + q(1 + r) \pm \sqrt{(s^2 + q(1 + r))^2 - 4 q s^2} . \quad (22)
$$

With (22) the solution of the coupled equations is given as (see (15) and (17))

$$
u_1 = u_1^0 \exp[-i\omega t + i k_1 x_1]] + u_1^0 \exp[-i\omega t - i k_1 x_1]]
+ \frac{m k_2}{s^2 - k_2^2} \{T_1^0 \exp[-i\omega t + i k_2 x_1]] + T_2^0 \exp[-i\omega t - i k_2 x_1]]\};
$$

$$
T = T_1^0 \exp[-i\omega t + i k_2 x_1]] + T_1^0 \exp[-i\omega t - i k_2 x_1]]
+ \frac{\eta k q i k_1}{k_1^2 - q} \{u_1^0 \exp[-i\omega t + i k_1 x_2]]
+ u_1^0 \exp[-i\omega t - i k_1 x_1]]\}, \quad (23)
$$
where $u_{+}, \ldots, T_{0}$ are the amplitudes of the waves propagating in the $+x_{1}$ and $-x_{1}$ directions with the frequency $\omega$.

We now analyze the roots $k_{1}$ and $k_{2}$ of equation (20). First, transform (20) into the form (see Nowacki (1975))

$$ (\xi^{2} - \chi^{2})(i\xi^{2} + \chi) + r\xi^{2} = 0, \quad (24) $$

where

$$ \xi = \frac{c_{2}}{\omega^{*}}k; \quad \omega^{*} = \frac{c_{1}^{2}}{\kappa}; \quad \chi = \frac{\omega}{\omega^{*}}, \quad (25) $$

and, as before,

$$ c_{1}^{2} = \frac{\mu}{\rho_{0}}; \quad c_{2}^{2} = \frac{\lambda + 2\mu}{\rho_{0}}. \quad (26) $$

The constant $\omega^{*}$ is called the characteristic frequency and is the property of the material. With the notation introduced, the solutions of (24) are

$$ \xi_{1} = \frac{1}{2} \sqrt{\chi} \left[ \sqrt{\chi + \sqrt{2\chi} + i(1 + r + \sqrt{2\chi})} \right. $$

$$ + \sqrt{\chi - \sqrt{2\chi} + i(1 + r - \sqrt{2\chi})} \right]; $$

$$ \xi_{2} = \frac{1}{2} \sqrt{\chi} \left[ \sqrt{\chi + \sqrt{2\chi} + i(1 + r + \sqrt{2\chi})} \right. $$

$$ + \sqrt{\chi - \sqrt{2\chi} + i(1 + r - \sqrt{2\chi})} \right]. \quad (27) $$

The values of $\xi_{1}, \xi_{2}$ depend on two parameters, $r$ and $\chi$. The parameter $r$ depends on the material and $\chi$ depends on the frequency $\omega$. In conclusion we may say, on the basis of (23), that in the case of longitudinal thermoelastic waves we have the modified elastic waves and modified thermal waves. Both elastic and thermal waves are damped and dispersed.

We consider finally a problem of wave propagation in an elastic half space. Thus, we consider a semi-infinite solid occupying the part $x_{2} \geq 0$ of the space. We assume that the waves are propagating in the direction of $x_{1}$ axis and that the component of the displacement vector $u_{3} = 0$. Then (1) becomes

$$ (\lambda + \mu) \frac{\partial^{2} u_{1}}{\partial x_{1}^{2}} + \mu \nabla^{2} u_{1} = \rho_{0} \frac{\partial^{2} u_{1}}{\partial t^{2}}; $$

$$ (\lambda + \mu) \frac{\partial^{2} u_{2}}{\partial x_{2}^{2}} + \mu \nabla^{2} u_{2} = \rho_{0} \frac{\partial^{2} u_{2}}{\partial t^{2}}, \quad (28) $$

where we assumed that $u_{i} = u_{i}(x_{1}, x_{2}), i = 1, 2$. The boundary conditions are taken in the form that corresponds to the stress-free plane $x_{2} = 0$; that is,

$$ \sigma_{ij} n_{j} = 0; \quad \text{at} \ x_{2} = 0. \quad (29) $$
The special type of solutions of the system (28), (29) was obtained by Rayleigh in 1887. It corresponds to the type of waves observed during earthquakes. Assume the solution of (28), (29) in the form

\[ u_1 = \hat{u}_1 + \bar{u}_1; \quad u_2 = \hat{u}_2 + \bar{u}_2, \]  

where \( \hat{u}_i \) and \( \bar{u}_i \), \( i = 1, 2 \) are solutions of (3) and (5), respectively;

\[ \mu \nabla^2 \hat{u}_i = \rho_0 \frac{\partial^2 \hat{u}_i}{\partial t^2}; \quad (\lambda + \mu) \nabla^2 \bar{u}_i = \rho_0 \frac{\partial^2 \bar{u}_i}{\partial t^2}. \]  

From (31) we conclude that the waves \( \hat{u}_i \) and \( \bar{u}_i \) propagate with velocities \( x_1 = \sqrt{\mu/\rho_0} \) and \( c_2 = \sqrt{(\lambda + 2\mu)/\rho_0} \), respectively. We take the solutions of (30) that decrease exponentially with \( x_2 \) so that

\[ \hat{u}_1 = C_1 \exp[-ax_2 + i(sx_1 - \omega t)]; \]
\[ \hat{u}_2 = C_2 \exp[-ax_2 + i(sx_1 - \omega t)]; \]
\[ \bar{u}_1 = D_1 \exp[-bx_2 + i(sx_1 - \omega t)]; \]
\[ \bar{u}_2 = D_2 \exp[-bx_2 + i(sx_1 - \omega t)], \]  

where \( C_1, C_2, D_1, D_2, a, b, s, \) and \( \omega \) are constants. The frequency \( \omega \) and the constant \( s \) determine the velocity of propagation as

\[ c_3 = \frac{\omega}{s}. \]  

By substituting (32) into (31) we obtain

\[ a^2 - s^2 + k^2 = 0; \quad b^2 - s^2 + h^2 = 0, \]  

where

\[ k^2 = \frac{\rho_0 \omega^2}{\mu}; \quad h^2 = \frac{\rho_0 \omega^2}{\lambda + 2\mu}. \]  

Since \( \hat{u}_i \) corresponds to shear waves the first invariant of the strain tensor calculated for \( \hat{u}_i \) must be equal to zero. Therefore

\[ \frac{\partial \hat{u}_1}{\partial x_1} + \frac{\partial \hat{u}_2}{\partial x_2} = 0. \]  

The condition (36) leads to

\[ C_2 ai = -C_1 s. \]  

Similarly since \( \bar{u}_i \) corresponds to an irrotational wave, we have

\[ \frac{\partial \bar{u}_2}{\partial x_1} - \frac{\partial \bar{u}_1}{\partial x_2} = 0, \]  

or

\[ D_2 is = -D_1 b. \]
Thus, there are only two independent constants and (32) can be written as

\[
\begin{align*}
\hat{u}_1 &= Ca \exp[-ax_2] \sin(sx_1 - \omega t); \\
\hat{u}_2 &= Cs \exp[-ax_2] \cos(sx_1 - \omega t); \\
\bar{u}_1 &= Ds \exp[-bx_2] \sin(sx_1 - \omega t); \\
\bar{u}_2 &= Db \exp[-bx_2] \cos(sx_1 - \omega t),
\end{align*}
\]

(40)

where \(C\) and \(D\) are arbitrary constants.

The boundary conditions express the fact that the plane \(x_2 = 0\) is stress free. From (29) it follows that

\[
\sigma_{12} = 0; \quad \sigma_{22} = 0 \quad \text{for} \quad x_2 = 0. \tag{41}
\]

From Hooke’s law and (40), (41) we obtain

\[
\mu \left( \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) = 0; \quad \lambda \left( \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \right) + 2\mu \frac{\partial u_1}{\partial x_1} = 0. \tag{42}
\]

By using (30) and (40) in (42) we obtain

\[
2\mu saC + [2\mu b^2 + \lambda (b^2 - s^2)]D = 0; \quad (a^2 + s^2)C + 2sbD = 0. \tag{43}
\]

There exists a nontrivial solution of (43) if

\[
[\lambda (b^2 - s^2) + 2\mu b^2] (a^2 + s^2) - 4\mu abs^2 = 0. \tag{44}
\]

However, from (34) we have

\[
a^2 = s^2 - k^2; \quad b^2 = s^2 - h^2. \tag{45}
\]

By using (45) in (44) we finally obtain

\[
\left( 2 - \frac{c_3^2}{c_1^2} \right)^4 - 16 \left( 1 - \frac{c_3^2}{c_2^2} \right) \left( 1 - \frac{c_3^2}{c_1^2} \right) = 0, \tag{46}
\]

where we used (26) and (33). The wave motion (40) describes the so-called surface or Rayleigh waves. The velocity of the surface waves is determined by solving (46). Special cases of surface wave velocity are:

i) Incompressible elastic material

In this case \(v = 1/2\) or \(\lambda \rightarrow \infty\) (see results of Problem 3 in Chapter 3). Therefore \(c_2 \rightarrow \infty\) and (46) becomes

\[
\left( 2 - \frac{c_3^2}{c_1^2} \right)^4 - 16 \left( 1 - \frac{c_3^2}{c_1^2} \right) = 0, \tag{47}
\]
From (47) we obtain from
\[
\left(\frac{c_3^2}{c_1^2}\right)^2 \left(\frac{c_3^2}{c_1^2}\right)^6 - 8 \left(\frac{c_3^2}{c_1^2}\right)^4 + 24 \left(\frac{c_3^2}{c_1^2}\right)^2 - 16 = 0. \tag{48}
\]

The real, nonzero solution of (48) is
\[
\frac{c_3^2}{c_1^2} = 0.91262, \tag{49}
\]
so that
\[
c_3 = 0.955 \sqrt{\frac{\mu}{\rho_0}}. \tag{50}
\]

ii) Poisson ratio equal to \( \nu = 1/4 \) (one constant elasticity)

In this case we have \( \lambda = \mu \) so that
\[
\frac{c_1^2}{c_2^2} = \frac{\mu}{\lambda + 2\mu} = \frac{1}{3}. \tag{51}
\]

We write next (46) in the form
\[
\left(2 - \frac{c_3^2}{c_1^2}\right)^4 - 16 \left(1 - \frac{c_3^2}{c_1^2}\right) \left(1 - \frac{c_3^2}{c_1^2}\right) = 0. \tag{52}
\]

Then, by using (51) in (52) it follows that
\[
3 \left(\frac{c_3^2}{c_1^2}\right)^6 - 24 \left(\frac{c_3^2}{c_1^2}\right)^4 + 56 \left(\frac{c_3^2}{c_1^2}\right)^2 - 32 = 0. \tag{53}
\]

There are three real roots of (53). They must lead to the real values of \( a \) and \( b \) in (34). From this condition it follows that
\[
\frac{c_3^2}{c_1^2} = 2 - 2\sqrt{3}, \tag{54}
\]
or
\[
c_3 = 0.9194 \sqrt{\frac{\mu}{\rho_0}}. \tag{55}
\]

By examining, numerically, the dependence of the roots of (46) on the Poisson ratio we conclude that
\[
c_3 = z \sqrt{\frac{\mu}{\rho_0}}, \tag{56}
\]
with \( z \in [0.874, 0.956] \) when \( \nu \in [0, 1/2] \).
We comment now on spherical waves. Suppose that the initial disturbance in an infinite *elastic* space is symmetric about the origin. Suppose further that the displacement vector is given by
\[ u_\varphi = u = \varphi(\rho, t); \quad u_\rho = 0; \quad u_\theta = 0. \] (57)

By using (57) in equations (4.3-13) and (4.3-14), we obtain

\[ \vartheta = 0; \quad \omega_\theta = -\frac{1}{2\pi} \frac{\partial}{\partial \rho}(\rho \varphi), \] (58)
\[ \frac{\mu}{\rho} \frac{\partial^2}{\partial \rho^2}(\rho \varphi) = \rho_0 \frac{\partial^2 \varphi}{\partial t^2}. \] (59)

The D'Alembert solution of (58) is
\[ \varphi = \frac{f_1(\rho - c_1 t)}{\rho} + \frac{f_2(\rho - c_1 t)}{\rho}, \] (60)

where \( c_1 = \sqrt{\frac{\mu}{\rho_0}} \) and \( f_1 \) and \( f_2 \) are arbitrary functions. The solution (59) represents two waves, one diverging from the source \( \rho = 0 \) with velocity \( c_1 \) and the other approaching the source with the same velocity.

**Problems**

1. Consider a prismatic rod with the cross-section shown in the figure.

The rod is loaded at its ends by two couples \( M \). Determine the stress field.

   a) First show that the Prandtl stress function \( \Psi \) can be written in the form
\[ \Psi = C \left( r^2 - a^2 - 2Ra \cos \theta + \frac{2Ra^2}{r} \cos \theta \right), \] (a)
where $C$ is a constant. In Cartesian coordinate system we have

$$x_2 = r \cos \theta; \quad x_3 = r \sin \theta; \quad r^2 = x_2^2 + x_3^2,$$

so that $(a)$ becomes

$$\Psi = C \left( x_2^2 + x_3^2 - a^2 - 2Rx_2 + \frac{2Ra^2x_2}{x_2^2 + x_3^2} \right).$$

By using $(a)$ in

$$\frac{\partial^2 \Psi}{\partial r^2} + \frac{1}{r} \frac{\partial \Psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Psi}{\partial \theta^2} = -2,$$

show that $C = -1/2$. Therefore the Prandtl stress function is

$$\Psi = \frac{1}{2} \left( x_2^2 + x_3^2 - a^2 - 2Rx_2 + \frac{2Ra^2x_2}{x_2^2 + x_3^2} \right)$$

$$= \frac{1}{2} \left( a^2 - r^2 + 2Rr \cos \theta - \frac{2Ra^2}{r} \cos \theta \right).$$

b) From $(e)$ determine the nonzero components of the stress tensor, as

$$\sigma_{r \theta} = \tilde{\sigma}_\mu \frac{1}{r} \frac{\partial \Psi}{\partial \theta} = -\tilde{\sigma}_\mu R \left[ 1 - \frac{a^2}{r^2} \right] \sin \theta;$$

$$\sigma_{\theta \theta} = -\tilde{\sigma}_\mu \frac{\partial \Psi}{\partial r} = \tilde{\sigma}_\mu \left[ r - R(1 + \frac{a^2}{r^2}) \cos \theta \right].$$

c) Show that the maximal value of the shear stress

$$\max \left| \sqrt{(\sigma_{r \theta})^2 + (\sigma_{\theta \theta})^2} \right| = \max p_\Psi = |\sigma_{\theta \theta}|_{\theta = 0} = \mu \tilde{\sigma}(2R - a).$$

Note that in the limit when $a \to 0$, we obtain

$$\max \left| \sqrt{(\sigma_{r \theta})^2 + (\sigma_{\theta \theta})^2} \right| = 2\mu \tilde{\sigma}R.$$ (h)

From $(h)$ we conclude that the maximal shear stress, in the limit $a \to 0$, is two times larger than the value of the maximal stress for the case of a perfectly circular rod (see (5.5-86) with $a = b = R$). Thus there is a stress concentration with the factor $k$ equal to two.

d) Show that the couple defined by (5.5-37),

$$M = 2\mu \tilde{\sigma} \int \int_A \Psi dA = 2\mu \tilde{\sigma} DR^4,$$ (i)
where

\[
D = \frac{1}{24} (\sin 4\alpha + 8 \sin 2\alpha + 12)\alpha - \frac{1}{2} \left(\frac{a}{R}\right)^2 (\sin 2\alpha + 2\alpha) + \frac{4}{3} \left(\frac{a}{R}\right)^3 \sin \alpha - \frac{1}{4} \left(\frac{a}{R}\right)^4 \alpha,
\]

and \(\cos \alpha = (a/2R)\).

2. For the prismatic rod with the cross-section in the shape of an equilateral triangle shown in the figure, determine the Prandtl stress function and the corresponding stress field if the rod is subjected to torsion couples at its ends.

![Equilateral triangle with labeled sides](image)

a) Show that the function

\[
\Psi = C \left[ x_2^2 + x_3^2 - \frac{(x_2^3 - 3x_2x_3^2)}{a} - \frac{4a^2}{27} \right],
\]

where \(C\) is a constant, satisfies the boundary conditions. Then by substituting (a) into \(\nabla^2 \Psi = -2\) show that \(C = -1/2\) so that

\[
\Psi = -\frac{1}{2} \left[ x_2^2 + x_3^2 - \frac{(x_2^3 - 3x_2x_3^2)}{a} - \frac{4a^2}{27} \right].
\]

b) Show that the shear stress components and the \(u_1\) component of the displacement vector are

\[
\sigma_{12} = -\mu \bar{\theta} \left( x_2 + \frac{3x_1x_2}{a} \right) ; \quad \sigma_{13} = \mu \bar{\theta} \left( x_2 + \frac{3}{2a} (x_2^2 - x_3^2) \right) ;
\]

\[
u_1 = \frac{\bar{\theta}}{2a} (3x_2x_3^2 - x_3^3).
\]

c) Finally show that the maximal value of the shear stress is at the middle point of the sides and has the value

\[
\max |\sqrt{(\sigma_{12})^2 + (\sigma_{13})^2}| = \max p_\Psi = \frac{1}{2} \mu \bar{\theta} a.
\]
3. Determine the deformation of an elastic sphere of radius $a$ if the body force in a spherical coordinate system is given as

$$f_\rho = -\rho_0 g \frac{\rho}{a}; \quad f_\theta = 0; \quad f_\phi = 0,$$  \hspace{1cm} (a)

where $\rho_0$ is the mass density of the sphere and $g$ is the acceleration of the "gravity" on the surface of the sphere. Thus show that

a) The equilibrium equations reduce to a single equation of the following form

$$\frac{d}{d \rho} \left( \frac{d u_\rho}{d \rho} + \frac{2 u_\rho}{\rho} \right) - \frac{\rho_0 g}{a(\lambda + 2\mu)} \rho = 0.$$  \hspace{1cm} (b)

The solution of (b) is

$$u_\rho = C \rho + \frac{\rho_0 g}{10a(\lambda + 2\mu)} \rho^3,$$  \hspace{1cm} (c)

where $C =$const.

b) By using the condition that the surface $\rho = a$ is stress free (see (3.6-2)), that is, $\sigma_{\rho\rho} = 0$ show that (c) becomes

$$u_\rho = -\frac{1}{10} \frac{\rho_0 ga}{\lambda + 2\mu} \rho \left[ \frac{5\lambda + 6\mu}{3\lambda + 2\mu} - \frac{\rho^2}{a^2} \right].$$  \hspace{1cm} (d)

Note that inside the sphere of radius

$$\rho = a \sqrt{\frac{3 - \nu}{3 + 3\nu}},$$  \hspace{1cm} (e)

the radial component of the strain tensor $E_{\rho\rho}$ is negative (corresponds to contraction), whereas outside this sphere $E_{\rho\rho}$ is positive (corresponds to extension).

4. Solve the system (5.12-6).

$$\frac{\partial \vartheta}{\partial \rho} + \frac{K}{\rho^2 \sin \varphi} \frac{\partial}{\partial \varphi} (\omega \sin \varphi) = 0; \quad \frac{\partial \vartheta}{\partial \varphi} - K \frac{\partial \omega}{\partial \rho} = 0.$$  \hspace{1cm} (a)

Thus start with the assumption there exists a function $\Phi$ of $\varphi$ and $\rho$ such that

$$\vartheta = K \frac{\partial \Phi}{\partial \rho}; \quad \omega = \frac{\partial \Phi}{\partial \varphi}.$$  \hspace{1cm} (b)

a) By substituting (b) into (a) show that $\Phi$ must be determined from

$$\frac{\partial^2 \Phi}{\partial \varphi^2} + \cot \varphi \frac{\partial \Phi}{\partial \varphi} + \rho^2 \frac{\partial^2 \Phi}{\partial \rho^2} = 0.$$  \hspace{1cm} (c)
and with $\Phi = R(\rho)F(\varphi)$ obtain
\[
\frac{d^2 R}{d\rho^2} - \frac{k(k + 1)}{\rho^2} R = 0; \quad \frac{d^2 F}{d\varphi^2} + \cot \varphi \frac{dF}{d\varphi} + k(k + 1)F = 0, \quad (d)
\]
where $k$ is an arbitrary integer.

b) Show that the solutions of $(d)$ are
\[
R_k = C_k \rho^{k+1} + D_k \rho^{-k}; \quad F_k = E_k P_k(s) + G_k Q_k(s), \quad (e)
\]
where $C_k$, $D_k$, $E_k$, and $G_k$ are constants, $s = \cos \varphi$, and $P_k$ and $Q_k$ are Legendre polynomials (see Lebedev (1972)). Therefore the general solution of $(c)$ is
\[
\Phi = \sum_{k=1}^{\infty} R_k F_k. \quad (f)
\]

5. Consider a radial vibration of a sphere with radius $R$. Make the assumption about the displacement field in the following form
\[
u_\rho = \rho f(\rho) \cos(\omega t + \alpha); \quad u_\theta = u_\varphi = 0, \quad (a)
\]
where $\omega$ and $\alpha$ are constants and $f(\rho)$ is a function to be determined. Then

i) Use (4.3-11) to show that $f$ satisfies
\[
\frac{d^2 f}{d\rho^2} + \frac{4}{\rho} \frac{df}{d\rho} + k^2 f = 0; \quad k^2 = \frac{\rho_0 \omega^2}{\lambda + 2\mu}. \quad (b)
\]
Then show that the solution of $(b)$ that is finite at $\rho = 0$ is
\[
f = \frac{C}{\rho^3} (k \rho \cos k \rho - \sin k \rho). \quad (c)
\]
Using Hooke's law in the form (3.6-2) show that
\[
\sigma_{\rho \rho} = \left[ (3\lambda + 2\mu)f + (\lambda + 2\mu)\rho \frac{df}{d\rho} \right] \cos(\omega t + \alpha). \quad (d)
\]
Then use the boundary condition
\[
\sigma_{\rho \rho}(t, \rho = R) = 0, \quad (e)
\]
to obtain the frequency equation
\[
(\lambda + 2\mu)[(2 - k^2 R^2) \sin kR - 2kR \cos kR]
+ 2\lambda(kR \cos kR - \sin kR) = 0. \quad (f)
\]
Equation $(f)$ determines $k$ and through $(b)$ the frequency $\omega$. 
6. By using (5.11-9) show that the displacement field in an elastic half space loaded by distributed tangential forces as shown in the figure are

\[
\begin{align*}
    u_1 &= \frac{1}{4\pi\mu} \int_{\Omega} \int \left\{ \frac{1}{\hat{\tau}} + \frac{\hat{x}_1^2}{\hat{\tau}^3} + (1 - 2\nu) \times \left[ \frac{1}{\hat{\tau} + x_3} - \frac{\hat{x}_1^2}{\hat{\tau}(\hat{\tau} + x_3)^2} \right] \right\} q(\xi, \eta) \, d\xi \, d\eta; \\
    u_2 &= \frac{1}{4\pi\mu} \int_{\Omega} \int \left\{ \frac{\hat{x}_1 \hat{x}_2}{\hat{\tau}^3} - (1 - 2\nu) \frac{\hat{x}_1 \hat{x}_2}{\hat{\tau}(\hat{\tau} + x_3)^2} \right\} d\xi \, d\eta; \\
    u_3 &= \frac{1}{4\pi\mu} \int_{\Omega} \int \left\{ \frac{\hat{x}_1 x_3}{\hat{\tau}^3} + (1 - 2\nu) \frac{\hat{x}_1}{\hat{\tau}(\hat{\tau} + x_3)} \right\} d\xi \, d\eta, \quad (a)
\end{align*}
\]

where

\[
\begin{align*}
    \hat{x}_1 &= x_1 - \xi; \quad \hat{x}_2 = x_2 - \eta; \\
    \hat{\tau} &= [(x_1 - \xi)^2 + (x_2 - \eta)^2 + x_3^2]^{1/2}. \quad (b)
\end{align*}
\]

It could be shown (see Johnson (1987)) that for the case when \( \Omega \) is a circle of radius \( a \) and when we take

\[
q(\xi, \eta) = q_0 \frac{1}{\sqrt{1 - (r/a)^2}}; \quad r = \sqrt{\xi^2 + \eta^2}, \quad (c)
\]

equations (a) for \( x_3 = 0 \) lead to

\[
\begin{align*}
    u_1 &= \frac{\pi(2 - \nu)}{4\mu} q_0 a; \quad u_2 = 0; \\
    u_3 &= -\frac{(1 - 2\nu)q_0 a}{2\mu} \left[ \frac{a}{r} - \frac{(s^2 - r^2)^{1/2}}{r} \right]. \quad (d)
\end{align*}
\]
7. Show that the components of the displacement vector for the Kelvin’s problem (5.12-8),(5.12-9) are given in the Cartesian coordinate system as

\[
\begin{align*}
    u_1 &= \frac{(\lambda + \mu)F_3}{8\pi\mu(\lambda + 2\mu)} \frac{x_1x_3}{r^3}; \\
    u_2 &= \frac{(\lambda + \mu)F_3}{8\pi\mu(\lambda + 2\mu)} \frac{x_2x_3}{r^3}; \\
    u_3 &= \frac{(\lambda + \mu)F_3}{8\pi\mu(\lambda + 2\mu)} \left( \frac{x_3^2}{r^3} + \frac{1}{r} \frac{\lambda + 3\mu}{\lambda + \mu} \right), \\
\end{align*}
\]

(a)

where \( r = (x_1^2 + x_2^2 + x_3^2)^{1/2} \). Expressions (a) agree with (5.9-5). Thus the stress field (5.12-9) in the Cartesian coordinate system is given by (5.9-16).
Chapter 6

Plane State of Strain and Plane State of Stress

6.1 Introduction

Generally speaking a solution of three-dimensional problems of elasticity theory is hard to obtain. It is therefore important to analyze a simpler case in which the quantities involved (stresses, strains, etc.) depend on two spatial coordinates only. In this class of problems belong the plane strain and so-called generalized plane stress problems that we now treat. The important characteristic of those simplified cases is that they could be solved by using the same mathematical formulation. The only difference is the values of certain constants in the solution. We treat both the static and dynamic problems.

We examine more closely the plane strain, plane stress, and generalized plane stress states. The first case, the plane strain problem, corresponds physically to the case of a long prismatic rod that is loaded on its lateral surface. Suppose that $x_3$ is oriented along the axis of the rod (see Fig. 7 of Chapter 2). We assume that the stress vector $\mathbf{p} = p_1\mathbf{e}_1 + p_2\mathbf{e}_2 + p_3\mathbf{e}_3$ on the surface of the rod is independent of the $x_3$ coordinate. The plane strain state corresponds to the case when the rod is infinitely long or when it has finite length but its ends are held in a special way. Then, the deformation tensor has the form studied in Section 2.8.

The plane stress and generalized plane stress problems correspond to the case of an elastic body of small thickness (plate) that is loaded by forces whose action lines are parallel to middle plane plates (spanned by $x_1, x_2$ coordinates), and the normal loads and stresses in the direction orthogonal to the plate ($x_3$ coordinate) are zero (see Fig. 1).

i) Plane strain state

As we saw in Section 2.8 in the case of plane strain the components of the displacement vector are

$$u_1 = u_1(x_1, x_2); \quad u_2 = u_2(x_1, x_2); \quad u_3 = 0 \text{ or } u_3 = \text{const.} \quad (1)$$

The strain tensor corresponding to (1) has the form

$$\mathbf{E} = \begin{bmatrix} E_{11} & E_{12} & 0 \\ E_{12} & E_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (2)$$
where
\[ E_{11} = \frac{\partial u_1}{\partial x_1}; \quad E_{22} = \frac{\partial u_2}{\partial x_2}; \quad E_{12} = \frac{1}{2} \left( \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right). \] (3)

All components \( E_{ij}, i, j = 1, 2 \) depend on \( x_1 \) and \( x_2 \) only. The components of the stress tensor are determined from (3.3-43) and read
\[ \sigma_{11} = \lambda \vartheta_1 + 2\mu \frac{\partial u_1}{\partial x_1}; \quad \sigma_{22} = \lambda \vartheta_1 + 2\mu \frac{\partial u_2}{\partial x_2}; \]
\[ \sigma_{33} = \lambda \vartheta_1; \quad \sigma_{12} = \mu \left( \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right); \quad \sigma_{13} = \sigma_{23} = 0, \] (4)

where
\[ \vartheta_1 = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}. \] (5)

Since \( E_{ij} \) depend on \( x_1 \) and \( x_2 \) it follows that components of the stress tensor depend on \( x_1 \) and \( x_2 \) and not on \( x_3 \). Using this, the equilibrium equations (1.3-4) become
\[ \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + f_1 = 0; \quad \frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + f_2 = 0. \] (6)

The third equation of the system (1.3-4) is identically satisfied since we assumed that the component of the body forces \( f_3 = 0 \). From (6) we conclude that \( f_1 \) and \( f_2 \) must be independent of \( x_3 \).

The present Lamé equations in the case for the dynamic problems are (see (4.1-6))
\[ (\lambda + \mu) \frac{\partial \vartheta_1}{\partial x_1} + \mu \Delta u_1 + \left( f_1 - \rho_0 \frac{\partial^2 u_1}{\partial t^2} \right) = 0; \]
\[ (\lambda + \mu) \frac{\partial \vartheta_1}{\partial x_2} + \mu \Delta u_2 + \left( f_2 - \rho_0 \frac{\partial^2 u_2}{\partial t^2} \right) = 0, \] (7)
where $\Delta$ is the two-dimensional Laplace operator, since all quantities are independent of $x_3$; that is, $\Delta(\cdot) = [\partial^2 / \partial x_1^2 + \partial^2 / \partial x_2^2](\cdot)$. The compatibility equations we take first in the form (2.7-14). For the strain tensor (2) they reduce to a single equation

$$\frac{\partial^2 E_{11}}{\partial x_2^2} + \frac{\partial^2 E_{22}}{\partial x_1^2} - 2\frac{\partial^2 E_{12}}{\partial x_1 \partial x_2} = 0. \quad (8)$$

If instead of Lamé equations (7) the equilibrium equations in the form (6) are used, the compatibility conditions of Beltrami-Michell must be used. Then (3.7-14) becomes

$$\frac{\partial^2 \Theta}{\partial x_1^2} + \frac{\partial^2 \Theta}{\partial x_2^2} = -\frac{1 - \nu}{1 + \nu} \left( \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \right). \quad (9)$$

To simplify (9) we determine the component $\sigma_{33}$ from Hooke’s law (3.4-43) with the additional condition $E_{33} = 0$. The result is

$$\sigma_{33} = \nu(\sigma_{11} + \sigma_{22}). \quad (10)$$

Thus, in the plane strain case, the number of independent stress components is three and $\sigma_{33}$ is independent of $x_3$. With the value for $\sigma_{33}$ given by (10), the first invariant of the stress tensor $\Theta = \sigma_{11} + \sigma_{22} + \sigma_{33}$ is

$$\Theta = (1 + \nu)(\sigma_{11} + \sigma_{22}). \quad (11)$$

By using (11) equation (9) becomes

$$\frac{\partial^2}{\partial x_1^2}(\sigma_{11} + \sigma_{22}) + \frac{\partial^2}{\partial x_2^2}(\sigma_{11} + \sigma_{22}) = -\frac{1}{1 - \nu} \left( \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \right). \quad (12)$$

Equation (12) could be derived directly from (8) as shown by Timoshenko and Goodier (1970).

The system of equations (4),(6),(8), and (12) constitutes the complete set of equations describing the plane state of strain. To these equations we must add the boundary conditions. They could be of different forms. For the first boundary value problem (see Chapter 4) the stress vector is prescribed on the boundary; that is,

$$\bar{p}_1 = \sigma_{11} n_1 + \sigma_{12} n_2; \quad \bar{p}_2 = \sigma_{12} n_1 + \sigma_{22} n_2; \quad \bar{p}_3 = 0, \quad (13)$$

where $\bar{p}_1$ and $\bar{p}_2$ are known functions defined for $(x_1, x_2)$ on the boundary of $C$ of the cross-section and $n$ is the unit outer normal on the boundary. In the case when (6) is solved, the boundary conditions (13) should be used directly. When Lamé equations (7) are used, it is necessary to substitute (4) into (13). The condition (13)$_3$ for the mixed boundary value problem when the length of the rod is finite is replaced by

$$u_3(x_1, x_2, 0) = u_3(x_1, x_2, l) = 0, \quad (14)$$
where \( l \) is the length of the rod. Thus, the plane strain problem is described by the system (4), (6), (8) and corresponding boundary conditions given by (6) and (13). In the case of dynamic problems the initial conditions must also be specified.

\emph{ii) Plane state of stress}

As we stated in Section 1.12 the plane state of stress is characterized by the condition that the stress vector on the planes that are parallel to the bases of the plate is zero throughout the plate

\[
\sigma_{13} = \sigma_{23} = \sigma_{33} = 0. \tag{15}
\]

With (15) the equilibrium equations take the form (6). However in general the nonzero components of the stress tensor, \( \sigma_{11}, \sigma_{12}, \) and \( \sigma_{22} \) are functions of \( x_1, x_2, \) and \( x_3. \) Therefore, the plane state of stress is a problem of three-dimensional elasticity.

Hooke's law in the case of plane state of stress leads to

\[
\begin{align*}
\sigma_{11} &= \lambda \vartheta + 2 \mu E_{11}; \\
\sigma_{22} &= \lambda \vartheta + 2 \mu E_{22}; \\
\sigma_{33} &= \lambda \vartheta + 2 \mu E_{33}; \\
\sigma_{12} &= 2 \mu E_{12}. \tag{16}
\end{align*}
\]

Using \( \sigma_{33} = 0 \) and solving (16)\textsubscript{3} for \( E_{33} \) we obtain

\[
E_{33} = -\frac{\lambda}{\lambda + 2 \mu} (E_{11} + E_{22}), \tag{17}
\]

so that

\[
\vartheta = E_{11} + E_{22} + E_{33} = \frac{2 \mu}{\lambda + 2 \mu} (E_{11} + E_{22}) = \frac{2 \mu}{\lambda + 2 \mu} \vartheta_1, \tag{18}
\]

where \( \vartheta_1 \) is given by (5). With this value substituted into Hooke's law (16) we obtain

\[
\begin{align*}
\sigma_{11} &= \frac{2 \mu \lambda}{\lambda + 2 \mu} \vartheta_1 + 2 \mu E_{11} = \frac{E}{1 - \nu^2} (E_{11} + \nu E_{22}); \\
\sigma_{22} &= \frac{2 \mu \lambda}{\lambda + 2 \mu} \vartheta_1 + 2 \mu E_{22} = \frac{E}{1 - \nu^2} (E_{22} + \nu E_{11}); \\
\sigma_{12} &= 2 \mu E_{12} = \frac{E}{1 + \nu} E_{12}. \tag{19}
\end{align*}
\]

From (4) and (19) we conclude that the expressions for the first three components of the stress tensor for the plane state of strain and for the plane state of stress are of the same type. It is only necessary to replace the constants \( \lambda, \mu \) and \( \nu \) by \( \lambda^*, \mu^*, \) and \( \nu^*; \) that is,

\[
\lambda \rightarrow \lambda^* = \frac{2 \mu \lambda}{\lambda + 2 \mu}; \quad \mu \rightarrow \mu^* = \mu; \quad \nu \rightarrow \nu^* = \frac{\nu}{1 + \nu}. \tag{20}
\]
For example, Lamé equations for the equilibrium of a body in a plane state of stress are (see (7))

\[(\lambda^* + \mu) \frac{\partial \theta_1}{\partial x_1} + \mu \Delta u_1 + f_1 = 0; \quad (\lambda^* + \mu) \frac{\partial \theta_1}{\partial x_2} + \mu \Delta u_2 + f_2 = 0. \quad (21)\]

and the equilibrium equations in terms of stresses are

\[\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + f_1 = 0; \quad \frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + f_2 = 0; \quad f_3 = 0. \quad (22)\]

However both components of the displacement vector and the stress tensor components, in general, depend on \(x_1, x_2,\) and \(x_3.\) That is why the plane stress problem is a three-dimensional elasticity problem. In Lamé equation (21) the operator \(\Delta\) is the three-dimensional Laplace operator; that is, \(\Delta(\cdot) = [\partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \partial^2/\partial x_3^2](\cdot).\)

There is one special case where the dependence on the variable \(x_3\) could be eliminated. It is called the "generalized plane stress problem" (see Love (1944)). Consider the plate shown in Fig. 1. Suppose that the external forces are distributed symmetrically with respect to the middle plane \(x_3 = 0.\) The boundary conditions on the upper and lower bases \(x_3 = \pm h\) are

\[\sigma_{31} = 0; \quad \sigma_{32} = 0; \quad \sigma_{33} = 0; \quad \text{for} \ x_3 = \pm h. \quad (23)\]

The third equilibrium equation (1.3-4) for a plate as a three-dimensional body reads

\[\frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} = 0, \quad (24)\]

where we used the assumption \(f_3 = 0.\) By using (23) in (24) we conclude that

\[\frac{\partial \sigma_{33}}{\partial x_3} = 0. \quad (25)\]

If we expand \(\sigma_{33}\) into a Taylor series with respect to \(x_3\) then, because of (24), (25), we have

\[\sigma_{33}(x_1, x_2, x_3) = \frac{1}{2} \frac{\partial^2 \sigma_{33}}{\partial x_3^2}(x_1, x_2, h)(x_3 - h)^2. \quad (26)\]

Therefore if we assume that the plate thickness is small, we have

\[\sigma_{33} = 0 \quad (27)\]

at every point of the plate. Due to the symmetry in loading, it follows that \(u_3, \sigma_{13}, \sigma_{23}\) are odd functions of the variable \(x_3\) so that the following holds: \(u_3(x_1, x_2, x_3) = -u_3(x_1, x_2, -x_3), \sigma_{13}(x_1, x_2, x_3) = -\sigma_{13}(x_1, x_2, -x_3),\) and \(\sigma_{23}(x_1, x_2, x_3) = -\sigma_{23}(x_1, x_2, -x_3).\) To eliminate dependence on \(x_3\) we introduce the mean values of the variables by the relations

\[u_i^m = \frac{1}{2h} \int_{-h}^{h} u_i dx_3; \quad \sigma_{ij}^m = \frac{1}{2h} \int_{-h}^{h} \sigma_{ij} dx_3; \quad f_i^m = \frac{1}{2h} \int_{-h}^{h} f_i dx_3, \quad (28)\]
for \( i, j = 1, 2, 3 \). It is easy to see that \( u_3(x_1, x_2, x_3) = -u_3(x_1, x_2, -x_3) \), so that \( u_3^m = 0 \). Similarly we conclude that \( \sigma^m_{13} = \sigma^m_{23} = 0 \). Then, multiplying the equilibrium equations

\[
\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} + f_1 = 0;
\]

\[
\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} + f_2 = 0;
\]

\[
\frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + f_3 = 0,
\]

by \( dx_3/2h \) and integrating between \([-h, h]\), we obtain

\[
\frac{\partial \sigma^m_{11}}{\partial x_1} + \frac{\partial \sigma^m_{12}}{\partial x_2} + f_1^m = 0; \quad \frac{\partial \sigma^m_{12}}{\partial x_1} + \frac{\partial \sigma^m_{22}}{\partial x_2} + f_2^m = 0.
\]

From (27) and Hooke’s law the relations (17) and (19) are obtained. Calculating the mean values of (19) it follows that

\[
\sigma_{11} = \frac{2\mu\lambda}{\lambda + 2\mu} \varphi^m_1 + 2\mu E^m_{11}
\]

\[
= \frac{E}{1 - (\nu^*)^2} \left( \frac{\partial u_1^m}{\partial x_1} + \frac{\partial u_2^m}{\partial x_2} \right);
\]

\[
\sigma_{22} = \frac{2\mu\lambda}{\lambda + 2\mu} \varphi^m_1 + 2\mu E^m_{22}
\]

\[
= \frac{E}{1 - (\nu^*)^2} \left( \frac{\partial u_2^m}{\partial x_2} + \frac{\partial u_1^m}{\partial x_1} \right);
\]

\[
\sigma_{12} = 2\mu E_{12}^m = \frac{E}{1 + \nu^*} \left( \frac{\partial u_1^m}{\partial x_2} + \frac{\partial u_2^m}{\partial x_1} \right).
\]

By comparing (30),(31) with (4),(6) we conclude that the mean (averaged over the thickness of the plate) components of the displacement vector and mean values of the components of the stress tensor satisfy the same equations as corresponding components in the plane state of strain, if we change the values of the constants according to (20).

### 6.2 Stress function method for the solution of plane problems

As we saw in the preceding section, the differential equations of equilibrium and Hooke’s law for the plane strain (and generalized plane stress) are of
the form
\[
\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + f_1 = 0; \quad \frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + f_2 = 0; \\
\]
\[
\sigma_{11} = \lambda \partial_1 + 2\mu E_{11} = \lambda \left( \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \right) + 2\mu \frac{\partial u_1}{\partial x_1}; \\
\sigma_{22} = \lambda \partial_1 + 2\mu E_{22} = \lambda \left( \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \right) + 2\mu \frac{\partial u_2}{\partial x_2}; \\
\sigma_{12} = 2\mu E_{12} = \mu \left( \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right). \tag{1}
\]

We assume that the cross-section of the body (middle plane in Fig. 1) is \(B\) with the contour \(C\). The compatibility equations are (6.1-8) or (6.1-12). Boundary conditions for the first boundary value problem could be written in the form
\[
(p_n)_1 = \sigma_{11} n_1 + \sigma_{12} n_2 = \bar{p}_1; \quad (p_n)_2 = \sigma_{12} n_1 + \sigma_{22} n_2 = \bar{p}_2, \tag{2}
\]
where \(n\) is the unit outer normal on the part of the contour \(C\) where the stress vector is prescribed.

There are many methods to solve the system (1),(2). The most efficient one is the use of the Airy stress function. Suppose that the body forces are potential; that is, there exists a scalar function \(V = V(x_i)\), such that the body forces are given as
\[
f_1 = -\frac{\partial V}{\partial x_1}; \quad f_2 = -\frac{\partial V}{\partial x_2}. \tag{3}
\]
Suppose further that there exists a scalar function \(\Phi = \Phi(x_1, x_2)\) such that the components of the stress could be expressed as
\[
\sigma_{11} = V + \frac{\partial^2 \Phi}{\partial x_2^2}; \quad \sigma_{22} = V + \frac{\partial^2 \Phi}{\partial x_1^2}; \quad \sigma_{12} = -\frac{\partial^2 \Phi}{\partial x_1 \partial x_2}. \tag{4}
\]
By substituting (4) into (1) we conclude that (1)\(_{1,2}\) are identically satisfied. The compatibility equation in the form (6.1-12) leads to
\[
\Delta \Delta \Phi = \nabla^4 \Phi = \frac{\partial^4 \Phi}{\partial x_1^4} + 2\frac{\partial^4 \Phi}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4 \Phi}{\partial x_2^4} = -\frac{1 - 2\nu}{1 - \nu} \left( \frac{\partial^2 V}{\partial x_1^2} + \frac{\partial^2 V}{\partial x_2^2} \right), \tag{5}
\]
where \(\Delta\) is the two-dimensional Laplace operator. In the case of plane stress the coefficient \((1 - 2\nu)/(1 - \nu)\) on the left-hand side of (5) changes, in accordance with (6.1-20) and becomes \((1 - \nu)\). Since the components of the unit vector \(n\) are
\[
n_1 = \cos \angle(n, \bar{x}_1) = \frac{dx_2}{dS}; \quad n_2 = \cos \angle(n, \bar{x}_2) = \frac{-dx_1}{dS}, \tag{6}
\]
it follows that equations (2) could be transformed by introducing (6) and (4). If we further assume that \( V = 0 \), then (2) becomes

\[
\frac{\partial^2 \Phi}{\partial x_1^2} dx_2 + \frac{\partial^2 \Phi}{\partial x_1 \partial x_2} dx_1 = \bar{p}_1 dS;
\]
\[
\frac{\partial^2 \Phi}{\partial x_2^2} dx_1 + \frac{\partial^2 \Phi}{\partial x_1 \partial x_2} dx_2 = -\bar{p}_2 dS.
\]  

(7)

From (7) it follows that

\[
\frac{\partial}{\partial x_1} \left[ \frac{\partial \Phi}{\partial x_2} \right] dx_1 + \frac{\partial}{\partial x_2} \left[ \frac{\partial \Phi}{\partial x_2} \right] dx_2 = d \left[ \frac{\partial \Phi}{\partial x_2} \right] = \bar{p}_1 dS;
\]
\[
\frac{\partial}{\partial x_1} \left[ \frac{\partial \Phi}{\partial x_1} \right] dx_1 + \frac{\partial}{\partial x_2} \left[ \frac{\partial \Phi}{\partial x_1} \right] dx_2 = d \left[ \frac{\partial \Phi}{\partial x_1} \right] = -\bar{p}_2 dS.
\]  

(8)

Integration of (8) along \( C \) leads to

\[
\frac{\partial \Phi}{\partial x_1} = - \int_0^S \bar{p}_2(u) du = \psi_1(S) + C_1; \quad \frac{\partial \Phi}{\partial x_2} = \int_0^S \bar{p}_1(u) du = \psi_2(S) + C_2.
\]  

(9)

where \( \psi_1(S) \) and \( \phi_2(S) \) are known functions and \( C_1 \) and \( C_2 \) are arbitrary constants.

The first boundary value problem for plane problems with simple connected regions and with zero body forces \( (V = 0) \) can now be stated as:

find \( \Phi \) that satisfies (5) with \( V = 0 \) in \( B \) and (9) on \( C \).

If the displacement components are prescribed on part (or the whole) of \( C \), we have to use Hooke’s law in the form solved for components of the strain tensor. From (6.1-4) we obtain

\[
\frac{\partial u_1}{\partial x_1} = \frac{1}{2\mu} [\sigma_{11} - \nu(\sigma_{11} + \sigma_{22})];
\]
\[
\frac{\partial u_2}{\partial x_2} = \frac{1}{2\mu} [\sigma_{22} - \nu(\sigma_{11} + \sigma_{22})];
\]
\[
\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} = \frac{\sigma_{12}}{\mu}.
\]  

(10)

Introducing (4), the system (10) becomes

\[
\frac{\partial u_1}{\partial x_1} = \frac{1}{2\mu} \left[ \frac{\partial^2 \Phi}{\partial x_2^2} - \nu \Delta \Phi \right] = \frac{1}{2\mu} \left[ (1 - \nu) \Delta \Phi - \frac{\partial^2 \Phi}{\partial x_1^2} \right];
\]
\[
\frac{\partial u_2}{\partial x_2} = \frac{1}{2\mu} \left[ \frac{\partial^2 \Phi}{\partial x_1^2} - \nu \Delta \Phi \right] = \frac{1}{2\mu} \left[ (1 - \nu) \Delta \Phi - \frac{\partial^2 \Phi}{\partial x_2^2} \right];
\]
\[
\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} = -\frac{1}{\mu} \frac{\partial^2 \Phi}{\partial x_1 \partial x_2}.
\]  

(11)
Integrating (11) we obtain

\[ u_1 = \frac{1}{2\mu} \int \left[ (1 - \nu)\Delta - \frac{\partial^2}{\partial x_1^2} \right] \Phi(x_1, x_2) dx_1 + \xi_1(x_2); \]
\[ u_2 = \frac{1}{2\mu} \int \left[ (1 - \nu)\Delta - \frac{\partial^2}{\partial x_2^2} \right] \Phi(x_1, x_2) dx_2 + \xi_2(x_2). \] (12)

In (12) \( \xi_1 \) and \( \xi_2 \) are arbitrary functions that are determined from the boundary conditions prescribed on \( U_1 \) and \( U_2 \) and from the condition (11) on the second boundary value problem for the plane state reads: find \( \Phi \) that satisfies (5) with \( V = 0 \) in \( B \) and (12) on the part of \( C \) where the displacement vector is prescribed.

Many solutions of the biharmonic equation (equation (5) with \( V = 0 \)),

\[ \Delta \Delta \Phi = \nabla^4 \Phi = \frac{\partial^4 \Phi}{\partial x_1^4} + 2 \frac{\partial^4 \Phi}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4 \Phi}{\partial x_2^4} = 0, \] (13)

in the Cartesian coordinate system \( \bar{x}_1 - \bar{x}_2 \) are known. We list some of elementary solutions that could be easily verified:

i) Polynomial of the second order

\[ \Phi = a_2 x_1^2 + b_2 x_1 x_2 + c_2 x_2^2, \] (14)

satisfies (13) for all values of constants \( a_2, b_2, \) and \( c_2. \)

ii) Polynomial of the third order

\[ \Phi = a_3 x_1^3 + b_3 x_1^2 x_2 + c_3 x_1 x_2^2 + d_3 x_2^3, \] (15)

satisfies (13) for all values of constants \( a_3, b_3, \) and \( c_3, d_3. \)

iii) Polynomial of the fourth order

\[ \Phi = a_4 x_1^4 + b_4 x_1^3 x_2 + c_4 x_1^2 x_2^2 + d_4 x_1 x_2^3 + e_4 x_2^4, \] (16)

satisfies (13) if

\[ e_4 = -a_4 - \frac{1}{3}c_4. \] (17)

iv) Polynomial of the fifth order

\[ \Phi = a_5 x_1^5 + b_5 x_1^4 x_2 + c_5 x_1^3 x_2^2 + d_5 x_1^2 x_2^3 + e_5 x_1 x_2^4 + f_5 x_2^5, \] (18)

satisfies (13) if

\[ e_5 = -5a_5 - c_5; \quad f_5 = -\frac{1}{5}(b_5 + d_5). \] (19)

v) Trigonometric and hyperbolic functions

\[ \chi \lambda x_1 \chi \lambda x_2; \quad \cos \lambda x_2 \chi \lambda x_1; \]
\[ \cos \lambda x_2 \chi x_1; \quad x_1 \cos \lambda x_1 \chi x_2, \] (20)
with $\lambda = \text{const.}$ satisfy (13).

vi) Linear combination of the functions (14) through (20) are solutions of (13). Also every potential function is a solution of (13).

vii) The solution to (13) could be obtained by the use of the separation of variables method; that is,

$$\Phi = f(x_1)g(x_2).$$

The assumption (21) leads to an infinite series of solutions of the type (20) that is given as

$$\Phi = \sum_{n=0}^{\infty} [(A_n + B_n x_2) \text{ch}(k_n x_2) + (C_n + D_n x_2) \text{ch}(k_n x_2)]$$

$$\times [C_{1n} \cos(k_n x_1) + C_{2n} \sin(k_n x_1)],$$

where $A_n, B_n, k_n, C_n, D_n, C_{1n},$ and $C_{2n}$ are constants that have to be determined from the boundary conditions.

We turn now to the problem (13) in a cylindrical coordinate system. By using

$$\Delta(\cdot) = \nabla^2 (\cdot) = \left( \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) (\cdot),$$

we obtain

$$\Delta \Delta \Phi = \nabla^4 \Phi = \left( \frac{\partial^4 \Phi}{\partial r^4} + \frac{2}{r} \frac{\partial^3 \Phi}{\partial r^3} - \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r^3} \frac{\partial \Phi}{\partial r} \right)$$

$$+ \left( \frac{2}{r^2} \frac{\partial^4 \Phi}{\partial r^2 \partial \theta^2} - \frac{2}{r^3} \frac{\partial^3 \Phi}{\partial r^3 \partial \theta^2} \right) + \frac{1}{r^4} \left( \frac{\partial^4 \Phi}{\partial \theta^4} + \frac{4}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2} \right) = 0.$$ \hspace{1cm} (24)

The components of the stress tensor are

$$\sigma_{rr} = \frac{1}{r} \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2}; \quad \sigma_{\theta \theta} = \frac{\partial^2 \Phi}{\partial r^2}; \quad \sigma_{r\theta} = -\frac{\partial}{\partial r} \left[ \frac{1}{r} \frac{\partial \Phi}{\partial \theta} \right]. \hspace{1cm} (25)$$

For the case of axial symmetry we have $\Phi = \Phi(r)$ so that

$$\Delta \Delta \Phi = \frac{d^4 \Phi}{dr^4} + \frac{2}{r} \frac{d^3 \Phi}{dr^3} - \frac{1}{r^2} \frac{d^2 \Phi}{dr^2} + \frac{1}{r^3} \frac{d \Phi}{dr} = 0,$$ \hspace{1cm} (26)

and

$$\sigma_{rr} = \frac{d \Phi}{dr}, \quad \sigma_{\theta \theta} = \frac{d^2 \Phi}{dr^2}, \quad \sigma_{r\theta} = 0.$$ \hspace{1cm} (27)

We list now some solutions of (24) and (26).

viii) The function

$$\Phi = Cr^2 \cos(2\theta),$$

where $C$ is a constant, is a solution of (24). The corresponding stress components are given by (27).
6.3 Some solutions of the plane problems

ix) The function

\[ \Phi = C_1 \ln r + C_2 r^2 + C_3 r^2 \ln r + C_4, \quad (29) \]

where \( C_i, i = 1,2,3,4 \) are constants is a solution of (26).

x) The function

\[ \Phi = C_1 \cos(2\theta) + C_2 \sin(2\theta), \quad (30) \]

where \( C_1 \) and \( C_2 \) are constants is a solution of (24).

xi) A general class of solutions of equation (24) could be obtained in terms of harmonic functions \( \psi (\Delta \psi = 0 \) by using the substitution

\[ \Phi = r \psi \cos \theta; \quad \Phi = r \psi \sin \theta; \quad \Phi = r^2 \psi. \quad (31) \]

From (31) it follows that the solution of (24) is

\[ \Phi = A_0 + B_0 \theta + A \ln r + B r^2 \ln r + C r^2 + Dr\theta(K_1 \cos \theta + K_2 \sin \theta) + (A_1 r + B_1 r^3 + C_1 \frac{1}{r} + D_1 r \ln r)(\bar{K}_1 \cos \theta + \bar{K}_2 \sin \theta) \]

\[ + \sum_{m=2,3} \left( A_m r^m + B_m r^{m+2} + C_m r^{-m} + D_m r^{-m+2} \right)(K_{1m} \cos m\theta + D_{2m} \sin m\theta) \]

\[ + \sum_{m=2,3} r^m \left[ A_m \cos m\theta + B_m \sin m\theta + C_m \cos(m-2)\theta + D_m \sin(m-2)\theta \right], \quad (32) \]

where \( A_0, B_0, A, B, C, D, K_1, K_2, \bar{K}_1, ..., D_{2m} \) are constants. The terms with \( A_0 \) and \( B_0 \) have no influence on the stress components and could be omitted. The term with the coefficient \( D_1 \) has importance in the plane stress state (to achieve uniqueness for the components of the displacement vector).

The problem of determining \( \Phi \) that satisfies given boundary conditions is not an easy one. The method that, in principle, gives a solution of (13) with arbitrarily prescribed boundary conditions based on complex variable representation of the solution of biharmonic equation is treated in Section 6.4. In the next section, we present several solutions of the plane problem obtained by various, more or less, elementary methods.

6.3 Some solutions of the plane problems

In this section we present some solutions to the plane problems by solving the corresponding equations in Cartesian and polar coordinate systems.
1. Cartesian coordinate system

1.1 Cantilever loaded at the free end by concentrated force

Consider a cantilever as shown in Fig. 2. Let \( l \) be the length of the cantilever, \( h \) its height, and \( \delta \) its thickness. The end \( x_1 = l \) is built in and at the free end \( x_1 = 0 \) a concentrated force \( F \) is applied.

![Figure 2](image)

Suppose we take the function \( \Phi \) in the form of linear combinations of the elementary solutions (14) and (16),

\[
\Phi = b_2 x_1 x_2 + d_4 x_1 x_2^3. \tag{1}
\]

In (1) \( b_2 \) and \( d_4 \) are constants. With (1) and \( V = 0 \), we obtain from (6.1-4)

\[
\sigma_{11} = 6d_4 x_1 x_2, \quad \sigma_{22} = 0; \quad \sigma_{12} = -b_2 - 3d_4 x_2^2. \tag{2}
\]

On the upper and lower side (i.e., \( x_2 = \pm h/2 \)) the unit normal vector is \( \mathbf{n} = \pm \mathbf{e}_2 \), so that the stress vector becomes

\[
\mathbf{p}_2 = \sigma_{12} \mathbf{e}_1 + \sigma_{22} \mathbf{e}_2; \quad \mathbf{p}_{-2} = -\sigma_{12} \mathbf{e}_1 - \sigma_{22} \mathbf{e}_2. \tag{3}
\]

From (3) and the fact that the upper and lower sides of the cantilever are stress free, it follows that

\[
\sigma_{12} = \pm \left( b_2 + \frac{3}{4} d_4 h^2 \right) = 0; \quad \sigma_{22} = 0. \tag{4}
\]

The condition (4)_2 is identically satisfied (see (2)_2) and by using (4)_1 the constant \( b_2 \) can be determined as

\[
b_2 = -\frac{3}{4} d_4 h^2. \tag{5}
\]

At the free end we have \( \mathbf{n} = -\mathbf{e}_1 \) so that

\[
\mathbf{p}_{-1} = -\sigma_{11} \mathbf{e}_1 - \sigma_{12} \mathbf{e}_2. \tag{6}
\]
6.3 Some solutions of the plane problems

If we calculate \( p_{-1} \) for \( x_1 = 0 \), we obtain

\[
p_{-1} = (b_2 + 3d_4 x_2^2)e_2,
\]

so that

\[
\delta \int_{-h/2}^{h/2} (b_2 + 3d_4 x_2^2) dx_2 = -F,
\]

where \( \delta \) is the thickness of the cantilever and \( F = -Fe_2 \). From (5) and (8) the constants \( d_4 \) and \( b_2 \) are determined as

\[
d_4 = \frac{2F}{h^3\delta}; \quad b_2 = -\frac{3F}{2h\delta}.
\]

Using (9) in (8) and (2) the stress tensor components read

\[
\sigma_{11} = \frac{F}{I} x_1 x_2; \quad \sigma_{12} = \frac{F}{2I} \left( \frac{h^2}{4} - x_2^2 \right); \quad \sigma_{22} = 0,
\]

where \( I = \delta/h^3/12 \) is the axial moment of inertia of the cross-section. The expressions (10) coincide with the solution obtained by elementary methods (Bernoulli plane cross-section hypothesis).

We determine now the components of the displacement vector. From (3.4-14) it follows that

\[
E_{11} = \frac{F}{EI} x_1 x_2; \quad E_{22} = -\frac{\nu F}{EI} x_1 x_2; \quad E_{12} = \frac{(1 + \nu)F}{2EI} \left( \frac{h^2}{4} - x_2^2 \right).
\]

With (11) and (2.9-1) we obtain

\[
\frac{\partial u_1}{\partial x_1} = \frac{F}{EI} x_1 x_2; \quad \frac{\partial u_2}{\partial x_2} = -\frac{\nu F}{EI} x_1 x_2,
\]

or, by integration

\[
u = \frac{F}{EI} \left[ \frac{1}{2} x_1^2 x_2 + f(x_2) \right]; \quad u_2 = -\frac{\nu F}{EI} \left[ \frac{1}{2} x_1 x_2^2 + \phi(x_1) \right],
\]

where \( f \) and \( \phi \) are arbitrary functions of \( x_2 \) and \( x_1 \), respectively. If we use (13) in (11) and (see comment after (6.2-12)) we obtain

\[
\frac{x_1^2}{2} + \frac{df}{dx_2} - \nu \left( \frac{x_2^3}{2} + \frac{d\phi}{dx_1} \right) = (1 + \nu) \left( \frac{h^2}{4} - x_2^2 \right),
\]

or

\[
\left[ \frac{df}{dx_2} \right] + \left[ \left( \frac{1 + \nu}{2} \right) x_2^2 \right] + \left[ \frac{x_1^2}{2} - \nu \frac{d\phi}{dx_1} \right] = \frac{1}{4} (1 + \nu) h^2.
\]

By solving (15) we get

\[
\frac{df}{dx_2} + \left( \frac{1 + \nu}{2} \right) x_2^2 = C; \quad \frac{(1 + \nu) h^2}{4} - \left[ \frac{x_1^2}{2} - \nu \frac{d\phi}{dx_1} \right] = C.
\]
where $C$ is a constant. Integrating (16) we finally have $f$ and $\phi$ in the form

$$f = Cx_2 - \frac{1}{3}x_2^3 \left(1 + \frac{\nu}{2}\right) + D; \quad \phi = \frac{1}{\nu} \left[Cx_1 + \frac{x_1^3}{5} - \frac{1}{4}(1 + \nu)h^2x_1 + K\right],$$

(17)

where $D$ and $K$ are constants. With (17) the components of the displacement vector (13) become

$$u_1 = \frac{F}{EI} \left[\frac{1}{2}x_1^2x_2 + Cx_2 - \frac{1}{3}x_2^3 \left(1 + \frac{\nu}{2}\right) + D\right];$$

$$u_2 = -\frac{F}{EI} \left[\frac{\nu}{2}x_1x_2^2 + x_1^3 + x_1 \left(C - \frac{1 + \nu}{4}h^2\right) + K\right].$$

(18)

The constants $C$, $D$, and $K$ we determine from the boundary conditions at $x_1 = l$. Suppose that the point $T$ (centroid of the cross-section at the right-hand end) is fixed; that is,

$$u_1(l, 0) = 0; \quad u_2(l, 0) = 0,$$

(19)

and that an element of the $\bar{x}_1$ axis is fixed (tangent to the line that in the undeformed state coincides with the $\bar{x}_1$ axis is tangent to the same line in the deformed state); that is,

$$\frac{\partial u_2}{\partial x_1} = 0; \quad \text{for } x_1 = l, \quad x_2 = 0.$$  

(20)

With (19), (20) substituted in (18) we have

$$D = 0; \quad C = \frac{(1 + \nu)h^2}{4} - \frac{l^2}{2}; \quad K = \frac{l^3}{3}.$$  

(21)

Finally by using (21) in (18) it follows that

$$u_1 = \frac{F}{EI} \left[\frac{x_1^2x_2}{2} + x_2 \left(\frac{1 + \nu}{4}h^2 - \frac{l^2}{2}\right) - \frac{x_2^3}{3} \frac{1 + \nu/2}{3}\right];$$

$$u_2 = -\frac{F}{EI} \left[\frac{\nu}{2}x_1x_2^2 + \frac{x_1^3}{6} - \frac{x_1l^2}{2} + \frac{l^3}{3}\right].$$

(22)

To examine the deformation closer, we fix a cross section of the cantilever at $x_1 = a$. After application of force, the point whose coordinates were $(x_1 = a, x_2)$ has the $\bar{x}_1$ coordinate given as $x_1 + u_1$; that is,

$$x_1 = a + \frac{F}{EI} \left[\frac{a^2x_2}{2} + x_2 \left(\frac{1 + \nu}{4}h^2 - \frac{l^2}{2}\right) - \frac{x_2^3}{3} \frac{1 + \nu/2}{3}\right].$$

(23)

Since $x_1$ depends on $x_2^3$ it is obvious that the plane cross-sections in the undeformed state do not remain plane in the deformed state; that is, the
Bernoulli hypothesis does not hold. The vertical displacement of the free end is obtained from \( u_2 \) \((x_1 = 0, \ x_2 = 0)\), that is
\[
u_2(x_1 = 0, \ x_2 = 0) = -\frac{Fl^3}{3EI}. \tag{24}
\]
The expression (24) agrees with the result obtained by elementary consideration with the Bernoulli hypothesis. At the cross-section where the cantilever is fixed the component of the displacement vector \( u_1 \) is
\[
u_1(x_1 = l, \ x_2) = \frac{F}{EI} \left[ \frac{1 + \nu}{4} h^2 x_2 x_1^2 - x_2^3 \frac{1 + \nu/2}{3} \right], \tag{25}
\]
and the slope at the axis
\[
\frac{\partial \nu_1}{\partial x_2}(x_1 = l, \ x_2 = 0) = \frac{Fh^2}{4EI} (1 + \nu) = \frac{3F}{2h\mu}. \tag{26}
\]
In Fig. 2b we showed graphically the cross-section at the point where the cantilever is fixed.

For the case when the boundary conditions at \( x_2 = l \) are taken in the form (19) and
\[
\frac{\partial \nu_1}{\partial x_2} = 0; \quad \text{for } x_1 = l, \ x_2 = 0, \tag{27}
\]
instead of (20) we obtain
\[
u_1 = \frac{F}{EI} \left[ \frac{x_1^2 x_2}{2} - x_2 \frac{l^2}{2} - x_2^3 \frac{1 + \nu/2}{3} \right];
\]
\[
u_2 = -\frac{F}{EI} \left[ \frac{\nu x_1 x_2^2}{2} + \frac{x_1^3}{6} - x_1 \frac{l^2}{2} + \frac{l^3}{3} + \frac{h^2(1 + \nu)}{4}(l - x_1) \right]. \tag{28}
\]
The vertical displacement of the free end is now
\[
u_2(x_1 = 0, \ x_2 = 0) = -\frac{Fl^3}{3EI} - \frac{Flh^2}{8\mu I}. \tag{29}
\]
Equation (29) represents the known result from the strength of material theory.

Similar results are obtained for the case of a cantilever loaded by a uniformly distributed load at the upper end. In this case the boundary conditions are
\[
p_1 = \pm \sigma_{12} = 0; \quad x_2 = \pm h;
\]
\[
p_2 = \sigma_{22} = -q; \quad x_2 = \frac{h}{2}, \quad p_2 = \sigma_{22} = 0, \quad x_2 = -\frac{h}{2}. \tag{30}
\]
where \( q \) is the intensity of the distributed loads on the upper side of the cantilever. The stress function in this case could be taken in the form of (14), (15), and (18); that is,

\[
\Phi = a_2 x_1^2 + b_3 x_1^2 x_2 + d_3 x_2^3 + d_5 (x_1^2 x_2^2 - \frac{1}{5} x_2^5).
\]  
(31)

The constants \( a_2, \ldots, d_5 \) could be determined from the boundary conditions (see Problem 8 at the end of this section).

### 1.2 Half plane loaded by uniformly distributed load

Consider a half plane \( x_1 > 0 \) loaded by a uniformly distributed load of intensity \( p(x_2) \) per unit length of the plane boundary (see Fig. 3). We use the Fourier transform method (see (4.5-8)) to solve the biharmonic equation (6.2-13); that is

\[
\Delta^2 \Phi = \frac{\partial^4 \Phi}{\partial x_1^4} + 2 \frac{\partial^4 \Phi}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4 \Phi}{\partial x_2^4} = 0. \tag{32}
\]

We use \( \tilde{f}(\xi) \) to denote the Fourier transform of \( f(x) \).

Then

\[
\tilde{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \exp(i\xi x) dx, \tag{33}
\]
where \( i^2 = -1 \). If \( f(x) \) satisfies the Dirichlet conditions\(^1\) then at the points where it is continuous, we have the inverse transformation

\[
f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(\xi) \exp(-i\xi x) d\xi, \tag{34}
\]

for \( -\infty < x < \infty \). Also if \( d^k f(x)/dx^k \to 0 \) when \( x \to \pm \infty \) for the case when \( k = 0, 1, \ldots, r - 1 \), then

\[
\int_{-\infty}^{\infty} \frac{d^r f(x)}{dx^r} \exp(i\xi x) dx = (-i\xi)^r \int_{-\infty}^{\infty} f(x) \exp(i\xi x) dx. \tag{35}
\]

\(^1\)The function \( f(x) \) satisfies Dirichlet conditions on an interval \((0, a)\) if it has no more than a finite number of discontinuities and a finite number of maxima and minima on the interval \((0, a)\).
Multiplying (32) by $[\exp(i\xi x_2)]/\sqrt{2\pi}$ and integrating with respect to $x_2$, we obtain

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left( \frac{\partial^4 \Phi}{\partial x_1^4} + 2 \frac{\partial^2 \Phi}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4 \Phi}{\partial x_2^4} \right) \exp(i\xi x_2) dx_2 = 0. \tag{36}$$

By using (35) in (36) it follows that

$$\frac{1}{\sqrt{2\pi}} \left[ \frac{d^4}{dx_1^4} \int_{-\infty}^{\infty} \Phi \exp(i\xi x_2) dx_2 + 2(-i\xi)^2 \frac{d^2}{dx_1^2} \int_{-\infty}^{\infty} \Phi \exp(i\xi x_2) dx_2 \right.

+ \left. (-i\xi)^4 \int_{-\infty}^{\infty} \Phi \exp(i\xi x_2) dx_2 \right] = 0 \tag{37}$$

or

$$\frac{d^4 \tilde{\Phi}}{dx_1^4} - 2\xi^2 \frac{d^2 \tilde{\Phi}}{dx_1^2} + \xi^4 \tilde{\Phi} = 0, \tag{38}$$

where $\tilde{\Phi}$ is the Fourier transform of $\Phi$ with respect to the variable $x_2$. From the definition (6.2-4) we obtain the Fourier transforms of the stress components $\sigma_{11}$, $\sigma_{22}$, and $\sigma_{12}$ as

$$\tilde{\sigma}_{11} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial^2 \Phi}{\partial x_2^2} \exp(i\xi x_2) dx_2 = -\frac{1}{\sqrt{2\pi}} \xi^2 \tilde{\Phi};$$

$$\tilde{\sigma}_{22} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial^2 \Phi}{\partial x_1^2} \exp(i\xi x_2) dx_2 = \frac{1}{\sqrt{2\pi}} \frac{d^2 \tilde{\Phi}}{dx_1^2};$$

$$\tilde{\sigma}_{12} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{-\partial^2 \Phi}{\partial x_1 \partial x_2} \exp(i\xi x_2) dx_2 = \frac{i\xi}{\sqrt{2\pi}} \frac{d\tilde{\Phi}}{dx_1}. \tag{39}$$

We next solve equation (38). The roots of the characteristic equation are

$$r_1 = r_2 = |\xi|; \quad r_3 = r_4 = -|\xi|. \tag{40}$$

Therefore the solution of (38) reads

$$\tilde{\Phi} = (A + Bx_1) \exp(-|\xi|x - 1) + (C + Dx_1) \exp(|\xi|x_1), \tag{41}$$

where $A$, $B$, $C$, and $D$ are constants. Since we require that (41) is bounded as $x_1 \to \infty$, we conclude that

$$C = D = 0, \tag{42}$$

so that (41) becomes

$$\tilde{\Phi} = (A + Bx_1) \exp(-|\xi|x_1). \tag{43}$$
We determine next the constants $A$ and $B$ from the boundary conditions. From (34) and (39) it follows that
\[
\sigma_{11}(x_1, x_2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{\sigma}_{11}(x_1, \xi) \exp(-i\xi x_2) d\xi \\
= -\frac{1}{2\pi} \int_{-\infty}^{\infty} \xi^2 \bar{\Phi} \exp(-i\xi x_2) d\xi;
\]
\[
\sigma_{22}(x_1, x_2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{\sigma}_{22}(x_1, \xi) \exp(-i\xi x_2) d\xi \\
= \frac{1}{2\pi} \int_{-\infty}^{\infty} d^2 \bar{\Phi} \exp(-i\xi x_2) d\xi;
\]
\[
\sigma_{12}(x_1, x_2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{\sigma}_{12}(x_1, \xi) \exp(-i\xi x_2) d\xi \\
= \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{d\Phi}{dx_1} \exp(-i\xi x_2) d\xi. \tag{44}
\]

Since on the boundary $x_1 = 0$ the components of the stress tensor are known, we have
\[
\tilde{\sigma}_{11}(0, \xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sigma_{11}(0, x_2) \exp(i\xi x_2) dx_2; \tag{45}
\]
\[
= -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} p(x_2) \exp(i\xi x_2) dx_2 = -\bar{p}(\xi);
\]
\[
\tilde{\sigma}_{12}(0, \xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sigma_{12}(0, x_2) \exp(i\xi x_2) dx_2 = 0.
\]

By using (39) and (43) in (45), we get
\[
-\bar{p}(\xi) = -\frac{1}{\sqrt{2\pi}} \left[ -\frac{1}{\sqrt{2\pi}} \xi^2 (A + B x_1) \exp(-|\xi|x_1) \right]_{x_1=0};
\]
\[
0 = \frac{i\xi}{\sqrt{2\pi}} \{[-|\xi|(A + B x_1) + B] \exp(-|\xi|x_1) \} _{x_1=0}. \tag{46}
\]

From (46) it follows that
\[
A = \frac{\sqrt{2\pi}}{\xi^2} \frac{\bar{\Phi}(\xi)}{\bar{p}(\xi)}; \quad B = \frac{\sqrt{2\pi}}{\xi^2} |\xi| \bar{p}(\xi). \tag{47}
\]

With $A$ and $B$ given by (47) the function $\bar{\Phi}$ is determined and (44) becomes
\[
\sigma_{11} = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{p}(\xi)[1 + |\xi|x_1] \exp(-|\xi|x_1 - i\xi x_2) d\xi;
\]
\[
\sigma_{22} = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{p}(\xi)[1 + |\xi|x_1] \exp(-|\xi|x_1 - i\xi x_2) d\xi;
\]
\[
\sigma_{12} = -\frac{i x_1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{p}(\xi) \exp(-|\xi|x_1 - i\xi x_2) d\xi. \tag{48}
\]
The system (48) determines the stress field for arbitrary \( p(x_2) \). As a special case, consider the distributed load shown in Fig. 4, that is

\[
p(x_2) = \begin{cases} 
  p_0, & \text{for } -a \leq x_2 \leq a \\
  0, & \text{for } |x_2| > a 
\end{cases}, \quad (49)
\]

Then, from (45) we obtain

\[
\bar{p}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} p_0 \exp(i\xi x_2) dx_2 = \frac{\sqrt{2}}{\sqrt{\pi}} p_0 \frac{\sin(\alpha \xi)}{\xi}. \quad (50)
\]

By substituting (50) into (48) we have

\[
\sigma_{11} = \frac{-2p_0}{\sqrt{2\pi}} \int_{0}^{\infty} \frac{1 + \xi x_1}{\xi} \exp(-\xi x_1) \sin(\alpha \xi) \cos(x_2 \xi) d\xi 
= \frac{p_0}{2\pi} [2(\theta_1 - \theta_2) + \sin(2\theta_1) - \sin(2\theta_2)];
\]

\[
\sigma_{22} = \frac{-2p_0}{\sqrt{2\pi}} \int_{0}^{\infty} \frac{1 - \xi x_1}{\xi} \exp(-\xi x_1) \sin(\alpha \xi) \cos(x_2 \xi) d\xi 
= \frac{p_0}{2\pi} [2(\theta_1 - \theta_2) - \sin(2\theta_1) + \sin(2\theta_2)];
\]

\[
\sigma_{12} = \frac{-2p_0 x_1}{\pi} \int_{0}^{\infty} \exp(\xi x_1) \xi \sin(\alpha \xi) \cos(\xi x_2) d\xi 
= \frac{p_0}{2\pi} [\cos(2\theta_2) - \cos(2\theta_1)],
\]

where we introduced the following notation

\[
\theta_1 = \arctan \frac{x_2 - a}{x_1}, \quad \theta_2 = \arctan \frac{x_2 + a}{x_1}. \quad (52)
\]

The geometrical meaning of the angles \( \theta_1 \) and \( \theta_2 \) is shown in Fig. 4.

As a special case, consider the half plane loaded by a concentrated force. Then (see Amenzade (1979))

\[
p(x_2) = -F\delta(x_2), \quad (53)
\]
where $\delta(x_2)$ is the Dirac $\delta$ function (see Section 5.1) and $F$ is the intensity of the concentrated force. By using (53) in (45) it follows that

$$p(\xi) = \frac{F}{\sqrt{2\pi}}. \quad (54)$$

With this value (48) becomes

$$\sigma_{11} = -\frac{2Fx_1^3}{\pi(x_1^2 + x_2^2)^2}; \quad \sigma_{22} = -\frac{2Fx_1x_2^2}{\pi(x_1^2 + x_2^2)^2}; \quad \sigma_{12} = -\frac{2Fx_1^2x_2}{\pi(x_1^2 + x_2^2)^2}. \quad (55)$$

Note that the solution (55) is singular at the point $x_1 = x_2 = 0$. Therefore it is valid only outside a half circle of small radius $r = (x_1^2 + x_2^2)^{1/2}$. On the boundary of this circle it is necessary to apply a distributed force whose resultant is $F$. The radius $r$ could be arbitrarily small. Then, according to the Saint-Venant principle (55) is the solution for concentrated force. Using Hooke’s law and (55) we can determine the displacement vector. We leave this problem as an exercise.

The generalization in which the force is not perpendicular to the free surface is given in Love (1944) and Landau and Lifshitz (1987).

1.3 Moving concentrated force

Consider next the problem of a moving concentrated force acting on the plane boundary. Let $v$ be the velocity of the force relative to a fixed coordinate system. Suppose that at the time instant $t_0 = 0$ the force is applied at the point $x_2 = \xi$. Then, the boundary conditions read (see (30) and (53))

$$\sigma_{11}(x_2) = -F\delta(x_2 - b - vt); \quad \sigma_{12} = 0, \quad \text{for } x_1 = 0. \quad (56)$$

Also we require that $\sigma_{ij} \to 0$ as $(x_1^2 + x_2^2)^{1/2} \to \infty$ (as is the case for solution (55)). We use the dynamic form of Lamé equations (see (4.1-6) and (6.1-7)). In the present case (no body forces) we obtain

$$(\lambda + \mu) \frac{\partial \vartheta}{\partial x_1} + \mu \Delta u_1 - \rho_0 \frac{\partial^2 u_1}{\partial t^2} = 0; \quad (\lambda + \mu) \frac{\partial \vartheta}{\partial x_2} + \mu \Delta u_2 - \rho_0 \frac{\partial^2 u_2}{\partial t^2} = 0. \quad (57)$$

By using (3.4-5) in (57) we get

$$\frac{\partial \vartheta}{\partial x_1} + (1 - 2\nu)\Delta u_1 - \frac{(1 - 2\nu)\rho_0}{\mu} \frac{\partial^2 u_1}{\partial t^2} = 0; \quad \frac{\partial \vartheta}{\partial x_2} + (1 - 2\nu)\Delta u_2 - \frac{(1 - 2\nu)\rho_0}{\mu} \frac{\partial^2 u_2}{\partial t^2} = 0, \quad (58)$$

where, as usual, we use $\mu$ for the shear modulus and $\nu$ for Poisson’s ratio.

The solution of (58) is assumed in the form (Alexandrov and Kovalenko 1986, p.286)

$$u_1 = \frac{\partial \varphi}{\partial x_1} + \frac{\partial \psi}{\partial x_2}; \quad u_2 = \frac{\partial \varphi}{\partial x_2} - \frac{\partial \psi}{\partial x_1}. \quad (59)$$
where $\varphi = \varphi(x_1, x_2, t)$ and $\psi = \psi(x_1, x_2, t)$ are functions to be determined. Note that from (59) it follows that

$$
\Delta \varphi = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}; \quad \Delta \psi = \frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}.
$$

(60)

By substituting (59) into (58) we obtain

$$
c_1^2 \Delta \psi = \frac{\partial^2 \psi}{\partial t^2}; \quad c_2^2 \Delta \varphi = \frac{\partial^2 \varphi}{\partial t^2},
$$

(61)

where (see (5.14-4),(5.14-6))

$$
c_1^2 = \frac{\mu}{\rho_0}; \quad c_2^2 = \frac{\lambda + 2\mu}{\rho_0}.
$$

(62)

The constants $c_1$ and $c_2$ are used to denote equivoluminal (shear) and irrotational (dilatational) wave velocities, respectively.

We consider next the problem (61) in a moving coordinate system. Thus, we introduce transformation of coordinates as

$$
\hat{x}_1 = x_1, \quad \hat{x}_2 = x_2 - vt.
$$

(63)

Then the force is applied at the fixed point $\hat{x}_2 = b$. Note that with (63) we have

$$
\frac{\partial^2 \varphi}{\partial t^2} = v^2 \frac{\partial^2 \varphi}{\partial \hat{x}_2^2}; \quad \frac{\partial^2 \psi}{\partial t^2} = v^2 \frac{\partial^2 \psi}{\partial \hat{x}_2^2}.
$$

(64)

By using (64) the system (61) becomes

$$
\frac{\partial^2 \varphi}{\partial \hat{x}_1^2} + \beta^2 \frac{\partial^2 \varphi}{\partial \hat{x}_2^2} = 0; \quad \frac{\partial^2 \psi}{\partial \hat{x}_1^2} + \gamma^2 \frac{\partial^2 \psi}{\partial \hat{x}_2^2} = 0,
$$

(65)

where

$$
\beta^2 = 1 - \frac{v^2}{c_2^2}; \quad \gamma^2 = 1 - \frac{v^2}{c_1^2}.
$$

(66)

The boundary conditions corresponding to (65) follow from (56) and are

$$
\mu \left[ (1 + \gamma^2) \frac{\partial^2 \varphi}{\partial \hat{x}_2^2} - 2 \frac{\partial^2 \psi}{\partial \hat{x}_1 \partial \hat{x}_2} \right] = -F \delta(\hat{x}_2 - b);
$$

$$
(1 + \gamma^2) \frac{\partial^2 \psi}{\partial \hat{x}_2^2} + 2 \frac{\partial^2 \varphi}{\partial \hat{x}_1 \partial \hat{x}_2} = 0; \quad \text{for } \hat{x}_1 = 0.
$$

(67)

Also $\varphi \to 0$ and $\psi \to 0$ as $(\hat{x}_1^2 + \hat{x}_2^2)^{1/2} \to \infty$. We solve (65), (67) under the condition that $\beta^2 > 0; \gamma^2 > 0$, that is, for

$$
v < c_1 = \sqrt{\frac{\mu}{\rho_0}}.
$$

(68)
By applying the Fourier transform (33) with respect to $\hat{x}_2$ the system (65), (67) becomes

\[
\frac{d^2 \tilde{\varphi}}{d\hat{x}_1^2} - \beta^2 \xi^2 \tilde{\varphi} = 0; \quad \frac{d^2 \tilde{\psi}}{d\hat{x}_1^2} - \gamma^2 \xi^2 \tilde{\psi} = 0,
\]

(69)

and for $\hat{x}_1 = 0$

\[
\xi^2 (1 + \gamma^2) \tilde{\varphi} + 2i\xi \frac{d\tilde{\psi}}{d\hat{x}_2} = \frac{F}{\mu} \exp(i\xi);
\]

\[
-\xi^2 (1 + \gamma^2) \tilde{\psi} + 2i\xi \frac{d\tilde{\varphi}}{d\hat{x}_2} = 0.
\]

(70)

The solutions $\tilde{\varphi}$ and $\tilde{\psi}$ must tend to zero as $\hat{x}_1 \to \infty$. It is easy to see that the solution to (69), (70) satisfying this condition is

\[
\tilde{\varphi}(\hat{x}_1, \xi) = \frac{-F(1 + \gamma^2)}{4\mu\gamma\beta(1 - B)\xi^2} \exp(i\xi - |\xi|\beta\hat{x}_1);
\]

\[
\tilde{\psi}(\hat{x}_1, \xi) = \frac{-2iF\beta\text{sgn}(\xi)}{4\mu\gamma\beta(1 - B)\xi^2} \exp(i\xi - |\xi|\beta\hat{x}_1),
\]

(71)

where $B = (1 + \gamma^2)^2/(4\beta\gamma)$. Taking the inverse Fourier transform of (71) and by using (59) the components of the displacement vector could be calculated. For example, at the boundary of the plane $x_1 = 0$ we have

\[
u_1(0, \hat{x}_2) = \frac{F(1 - \gamma^2)}{4\pi\mu\gamma(1 - B)}[-\ln |b - \hat{x}_2| + d],
\]

(72)

where $d$ is an arbitrary constant (corresponding to the displacement at infinity). Note that for $B \to 1$ in (72) we have $u_1(0, \hat{x}_2) \to \infty$. We analyze this case separately. With $B = 1$ we have $(1 + \gamma^2)^2 = 4\beta\gamma$. From this equation it follows that

\[
(2 - \frac{v^2}{c_1^2})^4 = 16 \left(1 - \frac{v^2}{c_2^2}\right) \left(1 - \frac{v^2}{c_3^2}\right).
\]

(73)

Comparing (73) with the equation determining the speed of Rayleigh waves (5.14-46) we conclude that $u_1(0, \hat{x}_2) \to \infty$ when the velocity $v$ of the moving force reaches the velocity of the Rayleigh waves $c_3$. Note also that for the case when $c_2 > v > c_3$ the displacement (72) is in the direction opposite to the direction of the force $F$. For the case when $v = 0$ the expression (72) becomes

\[
u_1(0, \hat{x}_2) = \frac{F(1 - v)}{\pi\mu}[-\ln |b - \hat{x}_2| + d],
\]

(74)

The expression (74) does not give zero displacement at infinite distances. Therefore there is difficulty in the application of the result (74) to an infinite plate (see Love (1944, p. 211)).
2. Plane problems in cylindrical coordinate system

2.1 A segment of a circular ring under the action of couples

As the first example consider a segment of a circular ring as shown in Fig. 5. We assume that the thickness of the segment is $\delta$. The boundary conditions are

$$
\sigma_{rr} = \sigma_{r\theta} = 0; \quad \text{for } r = r_1; \quad \sigma_{rr} = \sigma_{r\theta} = 0; \quad \text{for } r = r_2;
$$

$$
\delta \int_{r_1}^{r_2} \sigma_{\theta\theta} dr = 0, \quad \delta \int_{r_1}^{r_2} \sigma_{\theta\theta} rdr = M. \quad (75)
$$

As the stress function we use (6.2-29). The stress tensor components are given by (6.2-27). Therefore, we have

$$
\sigma_{rr} = \frac{C_1}{r^2} + 2C_2 + 2C_3 \ln r + C_3; \quad \sigma_{r\theta} = -\frac{C_1}{r^2} + 2C_2 + 2C_3 \ln r + 3C_3. \quad (76)
$$

By using (76) in (75)$_{1,2}$ we obtain

$$
\frac{C_1}{r_1^2} + 2C_2 + 2C_3 \ln r_1 + C_3 = 0; \quad \frac{C_1}{r_2^2} + 2C_2 + 2C_3 \ln r_2 + C_3 = 0. \quad (77)
$$

Equations (76) and (75)$_3$, lead to

$$
\int_{r_1}^{r_2} \sigma_{\theta\theta} dr = \left[ C_3(1 + 2 \ln r_2) + 2C_2 + \frac{C_1}{r_2^2} \right] r_2
$$

$$
- \left[ C_3(1 + 2 \ln r_1) + 2C_2 + \frac{C_1}{r_1^2} \right] r_1 = 0. \quad (78)
$$

Finally by using (76) in (75)$_4$ we obtain

$$
\frac{M}{\delta} = -C_1 \ln \frac{r_2}{r_1} + C_2(r_2^2 - r_1^2) + C_3[r_2^2(1 + \ln r_2) - r_1^2(1 + \ln r_1)]. \quad (79)
$$
Note that in the case when (77) is satisfied, then (78) is also satisfied. Therefore, we can determine the constants $C_1$, $C_2$, and $C_3$ from (77) and (79). The result is

$$C_1 = \frac{4M}{K} r_1^2 r_2^2 \ln \frac{r_2}{r_1}; \quad C_3 = \frac{2M}{K} (r_2^2 - r_1^2);$$

$$C_2 = -\frac{M}{K} \left[ r_2^2 - r_1^2 + 2(r_2^2 \ln r_2 - r_1^2 \ln r_1) \right], \quad \text{(80)}$$

where $K$ is given by

$$K = \left[ \left( r_2^2 - r_1^2 \right)^2 - 4r_1^2 r_2^2 \left( \ln \frac{r_2}{r_1} \right)^2 \right] \delta. \quad \text{(81)}$$

In Fig. 6 we show the distribution of stresses across the segment. In drawing the figure, we took $r_2 = 2r_1$ and $\delta = 1$. On the figure the point where $\sigma_{\theta\theta} = 0$ is shown. It corresponds to $r = r^* = 1.44401r_1$. This radius defines the “neutral axis” of the cross-section.

2.2 An elastic disc loaded by a couple

Consider a disc shown in Fig. 7. The disc is loaded by a concentrate couple $M$ at its axis. On the outer edge of the disc the tangential stresses are applied so that

$$\sigma_{rr} = 0. \quad \text{(82)}$$

To determine the stresses we assume the stress function in the form given by (6.2-32)

$$\Phi = C_1 \theta. \quad \text{(83)}$$

From (6.2-25) and (83) we obtain

$$\sigma_{rr} = 0; \quad \sigma_{\theta\theta} = 0; \quad \sigma_{r\theta} = \frac{C_1}{r^2}. \quad \text{(84)}$$
The constant \( C_1 \) is determined from the global equilibrium condition

\[
M = [\sigma_{\tau\theta}2\pi r]r\delta = 2\pi\delta C_1, \tag{85}
\]

where \( \delta \) is the thickness of the plate. From (84) and (85) it follows that

\[
C_1 = \frac{M}{2\pi\delta}; \quad \sigma_{\tau\theta} = \frac{M}{2\pi\delta r^2}. \tag{86}
\]

2.3 An infinite plate with a circular hole (the Kirsch problem)

Consider an infinite thin plate with a circular hole of radius \( r = a \). Suppose that at the center of the hole a coordinate system is introduced with the axes \( \bar{x}_1 - \bar{x}_2 \). We assume that at the large distance from the hole (i.e., \( r = (x_1^2 + x_2^2)^{1/2} \rightarrow \infty \)) the stress state corresponds to uniform tension in the \( \bar{x}_1 \) direction. Thus we assume that in the system \( \bar{x}_1 - \bar{x}_2 \) the stress tensor has the form

\[
\sigma = \begin{bmatrix} \sigma_0 & 0 \\ 0 & 0 \end{bmatrix}, \tag{87}
\]

for \( r = (x_1^2 + x_2^2)^{1/2} \rightarrow \infty \).

The stress field (87) corresponds to the stress function \( \Phi_0 \) given by

\[
\Phi_0 = \frac{\sigma_0}{2} x_2^2 = \frac{\sigma_0}{2} (r \sin \theta)^2 = \frac{\sigma_0 r^2}{4} - \frac{\sigma_0 r^2}{4} \cos(2\theta). \tag{88}
\]

We want to find the stress function \( \Phi \) such that on the circle of radius \( r = a \) it satisfies (see (6.2-25))

\[
\sigma_{rr}(r = a, \theta) = \frac{1}{r} \partial r \Phi + \frac{1}{r^2} \partial \theta^2 \Phi = 0;
\]

\[
\sigma_{r\theta}(r = a, \theta) = -\frac{\partial}{\partial \theta} \left[ \frac{1}{r} \partial r \Phi \right] = 0, \tag{89}
\]
and

\[
\lim_{r \to \infty} \sigma_{rr}(r, \theta) = \frac{1}{r} \frac{\partial \Phi_0}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Phi_0}{\partial \theta^2} = \frac{\sigma_0}{2} + \frac{\sigma_0}{2} \cos(2\theta);
\]

\[
\lim_{r \to \infty} \sigma_{r\theta}(r, \theta) = -\frac{\partial}{\partial r} \left[ \frac{1}{r} \frac{\partial \Phi_0}{\partial \theta} \right] = -\frac{\sigma_0}{2} \sin(2\theta),
\]

(90)

with \( \Phi_0 \) given by (88). Let

\[
\Phi = \Phi_1(r) + f(r) \cos(2\theta).
\]

(91)

The first function \( \Phi_1 \) we take in the form (6.2-29), that is,

\[
\Phi_1 = C_1 \ln r + C_2 r^2 + C_3 r^2 \ln r + C_4.
\]

(92)

By substituting \( f(r) \cos 2\theta \) into (6.2-24) we obtain

\[
\frac{d^4 f}{dr^4} + \frac{2}{r} \frac{d^3 f}{dr^3} - \frac{9}{r^2} \frac{d^2 f}{dr^2} + \frac{9}{r^3} \frac{df}{dr} = 0.
\]

(93)

The differential equation (93) is of the Euler type, and its general solution reads

\[
f = B_1 r^2 + B_2 r^4 + \frac{B_3}{r^2} + B_4.
\]

(94)

Therefore by using (91),(92),(94) the function \( \Phi \) becomes

\[
\Phi = C_1 \ln r + C_2 r^2 + C_3 r^2 \ln r + C_4 + (B_1 r^2 + B_2 r^4 + \frac{B_3}{r^2} + B_4) \cos(2\theta).
\]

(95)

The stress field corresponding to (95) has terms that increase to infinity for \( r \to \infty \) if \( C_3 \) and \( B_2 \) are different from zero. Also the term \( C_4 \) does not influence stresses. Therefore we take those constants equal to zero. Then from (90) we conclude that

\[
C_2 = -B_1 = \frac{\sigma_0}{4}.
\]

(96)

With these values we obtain

\[
\Phi = C_1 \ln r + \frac{\sigma_0}{4} r^2 + (\frac{\sigma_0}{4} r^2 + \frac{B_3}{r^2} + B_4) \cos(2\theta).
\]

(97)

The boundary conditions (89) applied to (95) lead to

\[
C_1 = \frac{\sigma_0}{2} a^2; \quad B_3 = -\frac{\sigma_0}{4} a^2; \quad B_4 = \frac{\sigma_0}{2} a^2.
\]

(98)

Thus, the stress function reads

\[
\Phi = \frac{\sigma_0}{2} a^2 \ln r + \frac{\sigma_0}{4} r^2 + (\frac{\sigma_0}{4} r^2 - \frac{\sigma_0}{4} a^2 \frac{1}{r^2} + \frac{\sigma_0}{2} a^2) \cos(2\theta).
\]

(99)
With (99) the stress tensor components (6.2-25) become

\[
\sigma_{rr} = \frac{\sigma_0}{2} \left[ (1 - \frac{a^2}{r^2}) + (1 - 4\frac{a^2}{r^2} + 3\frac{a^4}{r^4}) \cos(2\theta) \right];
\]

\[
\sigma_{\theta\theta} = \frac{\sigma_0}{2} \left[ (1 - \frac{a^2}{r^2}) - (1 + 3\frac{a^4}{r^4}) \cos(2\theta) \right];
\]

\[
\sigma_{r\theta} = -\frac{\sigma_0}{2} \left[ (1 + 2\frac{a^2}{r^2} - 3\frac{a^4}{r^4}) \cos(2\theta) \right].
\]  \hfill (100)

From (100) it follows that the maximal stress is at the point \((r = a, \theta = \pi/2)\) and has the value

\[
\sigma_{r\theta}(r = a, \theta = \pi/2) = 3\sigma_0.
\]  \hfill (101)

The stress concentration factor due to the hole is therefore

\[
k = 3.
\]  \hfill (102)

We note that by similar arguments it could be shown that for the case of
an infinite plate with the hole and the stress state far from the hole given
in the form of biaxial tension

\[
\sigma = \begin{bmatrix}
\sigma_0 & 0 \\
0 & \sigma_0 
\end{bmatrix},
\]  \hfill (103)

the stress concentration factor is

\[
k = 2,
\]  \hfill (104)

whereas for the case of tension in one and compression in the other direction

\[
\sigma = \begin{bmatrix}
\sigma_0 & 0 \\
0 & -\sigma_0 
\end{bmatrix},
\]  \hfill (105)

the stress concentration factor is

\[
k = 4.
\]  \hfill (106)

6.4 Complex variable method for plane problems

As stated in Section 6.2 for the case of zero body forces, the solution of a
plane problem is known if the stress function \(\Phi\) satisfying

\[
\Delta\Delta\Phi = 0,
\]  \hfill (1)
We present a special procedure to find a solution of (1) by using complex variable method. Let \( z = x_1 + ix_2 \) be a complex \( (i = \sqrt{-1}) \) variable and \( \bar{z} = x_1 - ix_2 \) its complex conjugate. Note that

\[
x_1 = \frac{1}{2}(z + \bar{z}), \quad x_2 = \frac{1}{2}(z - \bar{z}).
\]

By using (2) it follows that the Laplace’s operator can be expressed as

\[
\Delta(\cdot) = \frac{\partial^2}{\partial x_1^2}(\cdot) + \frac{\partial^2}{\partial x_2^2}(\cdot) = 4\frac{\partial^2}{\partial z \partial \bar{z}}(\cdot).
\]

In Section 4.1 we showed (see (4.1-13)) that with zero body forces the first invariant of the strain tensor is a harmonic function

\[
\Delta \psi = \frac{\partial^2 \psi}{\partial x_1^2} + \frac{\partial^2 \psi}{\partial x_2^2} = \frac{\partial^2 \psi_1}{\partial x_1^2} + \frac{\partial^2 \psi_1}{\partial x_2^2} = \Delta \left( \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \right) = 0,
\]

where we used (6.1-5). If through (2) \( \psi \) is expressed in terms of \( z \) and \( \bar{z} \) the condition (4) becomes

\[
\Delta \psi_1 = \Delta \left( \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \right) = 4\frac{\partial^2 \psi_1}{\partial z \partial \bar{z}} = 0.
\]

By integrating (5) we obtain

\[
\psi_1 = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = \frac{1}{\lambda + \mu} \left[ \varphi'(z) + \overline{\varphi'(z)} \right],
\]

where \( \psi \) is an arbitrary analytic function of \( z \) and \( \overline{\psi} \) is its complex conjugate. The factor \( 1/(\lambda + \mu) \) in front of the right-hand side is chosen for convenience. Also if \( \psi(z) = u + iv \) is a function of \( z \) then

\[
\varphi' = \frac{\partial u(x_1, x_2)}{\partial x_1} + i \frac{\partial v(x_1, x_2)}{\partial x_1}.
\]

\(^2\) The functions \( u \) and \( v \) satisfy the Cauchy-Riemann conditions \( (\partial u/\partial x_1) = (\partial v/\partial x_2) \), \( (\partial u/\partial x_2) = -(\partial v/\partial x_1) \).
We turn now to Lamé equations (6.1-21) that in the absence of body forces read

\[(\lambda + \mu) \frac{\partial \theta_1}{\partial x_1} + \mu \Delta u_1 = 0; \quad (\lambda + \mu) \frac{\partial \theta_1}{\partial x_2} + \mu \Delta u_2 = 0.\]  

(8)

Multiplying (8)_2 by imaginary unity \(i\) and adding to (8)_1 we obtain

\[\mu \Delta(u_1 + iu_2) + (\lambda + \mu) \left( \frac{\partial \theta_1}{\partial x_1} + i \frac{\partial \theta_1}{\partial x_2} \right) = 0.\]

(9)

From (2)

\[\left( \frac{\partial \theta_1}{\partial x_1} + i \frac{\partial \theta_1}{\partial x_2} \right) = 2 \frac{\partial \theta_1}{\partial z},\]

(10)

so that by using (5) we transform (9) into

\[2\mu \frac{\partial^2(u_1 + iu_2)}{\partial z \partial \bar{z}} + (\lambda + \mu) \frac{\partial \theta_1}{\partial \bar{z}} = 0.\]

(11)

By integrating (11) it follows that

\[2\mu \frac{\partial(u_1 + iu_2)}{\partial z} + (\lambda + \mu) \theta_1 = \varphi_1(z),\]

(12)

where \(\varphi_1\) is an arbitrary analytic function. Taking the complex conjugate of (12) we obtain

\[2\mu \frac{\partial(u_1 - iu_2)}{\partial \bar{z}} + (\lambda + \mu) \theta_1 = \overline{\varphi_1(z)}.\]

(13)

For any analytic function \(f(z) = u + iv\) we have

\[\frac{\partial f(z)}{\partial z} + \frac{\partial \bar{f}}{\partial \bar{z}} = \frac{\partial(u + iv)}{\partial z} + \frac{\partial(u - iv)}{\partial \bar{z}} = \frac{\partial u}{\partial x_1} + \frac{\partial v}{\partial x_1}.\]

(14)

Then by adding (12) to (13) and by using (14) and (6) we obtain

\[2(\lambda + 2\mu) \theta_1 = \frac{2(\lambda + 2\mu)}{\lambda + \mu} [\varphi' + \bar{\varphi}'] = \varphi_1 + \bar{\varphi}_1.\]

(15)

From (15) it follows that the functions \(\varphi'\) and \(\varphi_1\) are connected as follows

\[\varphi_1(z) = \frac{2(\lambda + 2\mu)}{\lambda + \mu} \varphi'(z) + ic,\]

(16)

where \(c\) is an arbitrary real constant. Finally by using (6),(12), and (16) we obtain

\[2\mu(u_1 + iu_2) = \kappa \varphi(z) - z \overline{\varphi'(z)} - \overline{\varphi(z)} + icz,\]

(17)

where

\[\kappa = \frac{\lambda + 3\mu}{\lambda + \mu} = 3 - 4\nu,\]

(18)
and $\psi$ is an arbitrary analytic function. The term $icz$ in (17) corresponds to a rigid body displacement and could be neglected. Thus, from (17) we obtain

$$u_1 + iu_2 = \frac{1}{2\mu} [\kappa \varphi(z) - z \varphi'(z) - \psi(\bar{z})].$$

(19)

The formula (19) was derived by Kolossov in his 1909 dissertation (see Kolossov (1914)). For the case of the generalized plane stress state, (19) becomes

$$u_1 + iu_2 = \frac{1}{2\mu} [\kappa^* \varphi(z) - z \varphi'(z) - \psi(\bar{z})],$$

(20)

where

$$\kappa = \frac{\lambda^* + 3\mu}{\lambda^* + \mu} = \frac{3 - \nu}{1 + \nu}.$$  

(21)

We turn now to the problem of determining stresses. From the compatibility conditions (6.1-12) we have

$$\Delta \Theta = \frac{\partial^2}{\partial x_1^2} (\sigma_{11} + \sigma_{22}) + \frac{\partial^2}{\partial x_2^2} (\sigma_{11} + \sigma_{22}) = 0.$$  

(22)

By using (3) equation (22) becomes

$$\Delta \Theta = 4 \frac{\partial^2 \Theta}{\partial z \partial \bar{z}},$$

so that

$$\sigma_{11} + \sigma_{22} = 2[\psi_1(z) + \psi_1(\bar{z})],$$

(24)

where $\psi_1$ is an analytic function. However, from Hooke’s law we have (see (6.1-4))

$$\Theta = \sigma_{11} + \sigma_{22} = 2(\lambda + \mu) \psi_1.$$  

(25)

From (6), (24), and (25) we conclude that $\psi_1(z) = \varphi'(z)$. Therefore

$$\sigma_{11} + \sigma_{22} = 2[\varphi'(z) + \overline{\varphi'(z)}] = 4 \Re(\varphi'(z)),$$

(26)

where $\Re(\varphi'(z))$ denotes the real part of $\varphi'(z)$. We now write the equilibrium equations (6.1-6) in the complex form. Consider the equilibrium equations with zero body forces

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} = 0; \quad \frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} = 0.$$  

(27)
Multiply (27)\textsubscript{2} by \( i \) and subtract the result from (27)\textsubscript{1}. Then

\[
\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} - i \frac{\partial \sigma_{12}}{\partial x_1} - i \frac{\partial \sigma_{22}}{\partial x_2} = \frac{\partial}{\partial z} [\sigma_{11} - i \sigma_{12} + i \sigma_{12} + \sigma_{22}] + \frac{\partial}{\partial z} [\sigma_{11} - i \sigma_{12} + i \sigma_{12} - \sigma_{22}] = 0, \tag{28}
\]

or

\[
\frac{\partial}{\partial z} [\sigma_{11} + \sigma_{22}] = \frac{\partial}{\partial z} [\sigma_{22} - \sigma_{11} + 2i \sigma_{12}] . \tag{29}
\]

From (26) and (29) we obtain

\[
\sigma_{22} - \sigma_{11} + 2i \sigma_{12} = 2[z'\varphi''(z) + \psi'(z)], \tag{30}
\]

where \( \psi'(z) \) is an analytic function introduced by (19). By using (20), (26), and (30) we conclude that in plane problems two analytic functions \( \varphi(z) \) and \( \psi(z) \) completely determine stresses and displacement.

We derive now a complex function representation for solution of a biharmonic equation (6.2-5) with zero body forces. Namely, from (6.2-4)

\[
\sigma_{11} = \frac{\partial^2 \Phi}{\partial x_2^2}; \quad \sigma_{22} = \frac{\partial^2 \Phi}{\partial x_1^2}; \quad \sigma_{12} = -\frac{\partial^2 \Phi}{\partial x_1 \partial x_2} . \tag{31}
\]

By using (2), (3) we have

\[
\sigma_{11} + \sigma_{22} = \Delta \Phi = 4 \frac{\partial^2 \Phi}{\partial z \partial \bar{z}}; \quad \sigma_{22} - \sigma_{11} + 2i \sigma_{12} = 4 \frac{\partial^2 \Phi}{\partial z^2} . \tag{32}
\]

From (32), (26), and (30) we obtain

\[
2 \frac{\partial^2 \Phi}{\partial z \partial \bar{z}} = \varphi'(z) + \varphi'(z); \quad 2 \frac{\partial^2 \Phi}{\partial z^2} = z \varphi''(z) + \psi'(z). \tag{33}
\]

Equations (33) lead to

\[
2 \frac{\partial \Phi}{\partial \bar{z}} = \varphi(z) + z \varphi'(z) + g_1(z); \quad 2 \frac{\partial \Phi}{\partial \bar{z}} = z \varphi'(z) + \psi(z) + g_2(z) . \tag{34}
\]

By comparing (34)\textsubscript{1} and (34)\textsubscript{2} we find

\[
\varphi(z) - g_2(z) = \overline{\psi(z)} - g_1(z) , \tag{35}
\]

so that

\[
g_1(z) = \psi(z) + c; \quad g_2(z) = \varphi(z) + c , \tag{36}
\]

where \( c \) is a complex constant. Finally by using (36) in (34) we get

\[
2 \frac{\partial \Phi}{\partial \bar{z}} = \varphi(z) + z \varphi'(z) + \psi(z) + c , \tag{37}
\]
or
\[
\Phi = \frac{1}{2} \left[ \bar{z} \varphi(z) + z \bar{\varphi(z)} + \int \psi(z) d\bar{z} + \chi(z) + \bar{\chi}(z) \right],
\] (38)
with \( \chi(z) \) being another analytic function. Since \( \Phi \) and its second derivatives are real quantities it follows that
\[
\chi(z) = \int \psi(z) d\bar{z} + cz + c_0,
\] (39)
where \( c_0 \) is an arbitrary complex constant. Without loss of generality we take \( c = c_0 = 0 \). Then from (38),(39) we have
\[
\Phi = \frac{1}{2} \left[ \bar{z} \varphi(z) + z \bar{\varphi(z)} + \chi(z) + \bar{\chi}(z) \right] = \Re(\bar{z} \varphi(z) + \chi(z)).
\] (40)
By using (40) and (32) we obtain
\[
\sigma_{11} + \sigma_{22} = 2[\varphi'(z) + \bar{\varphi'(z)}] = 4\Re(\varphi'(z));
\]
\[
\sigma_{22} - \sigma_{11} + 2i\sigma_{12} = 2[\bar{z} \varphi''(z) + \psi'(z)],
\] (41)
in agreement with (26),(30). The expression (40) is the Goursat form of the solution of the biharmonic equation. For another derivation of (40) see Muskhelishvilli (1966).

We turn now to the problem of determining functions \( \varphi \) and \( \psi \) in (20),(26), and (30). We consider first the second boundary value problem (see Section 4.2).

The second fundamental boundary value problem: given the displacement vector \( U = U_1 e_1 + U_2 e_2 \) on curve \( C \) that is a boundary of \( B \), determine two analytic functions \( \varphi \) and \( \psi \) satisfying
\[
\tilde{u}_1 + i\tilde{u}_2 = \frac{1}{2\mu} \left[ \kappa^* \varphi(z) - z \varphi'(z) - \psi(z) \right],
\] (42)
on \( C \). In (42) \( \tilde{u}_1 \) and \( \tilde{u}_2 \) are known functions of the points on \( C \).

The first fundamental boundary value problem: given the components of the stress vector on the curve \( C \) representing the boundary of \( B \)
\[
\sigma_{11} n_1 + \sigma_{12} n_2 = \hat{p}_1; \quad \sigma_{12} n_1 + \sigma_{22} n_2 = \hat{p}_2,
\] (43)
with \( \hat{p}_1 \) and \( \hat{p}_2 \) known functions on \( C \). We define a complex stress vector
\[
\mathbf{p}_n = \hat{p}_1 + i\hat{p}_2.
\] (44)
Then by using \( n_1 = dx_2/dS; \ n_2 = -dx_1/dS \) (see (6.2-6)) and (6.2-4) we obtain from (43) and (44)
\[
\mathbf{p}_n = \hat{p}_1 + i\hat{p}_2 = \frac{\partial^2 \Phi}{\partial x_2^2} \frac{dx_2}{dS} - i\frac{\partial^2 \Phi}{\partial x_1^2} \frac{dx_1}{dS} = -i \frac{\partial}{\partial S} \left( \frac{\partial \Phi}{\partial x_1} + i \frac{\partial \Phi}{\partial x_2} \right).
\] (45)
From (45) it follows that

\[
\frac{\partial \Phi}{\partial x_1} + i \frac{\partial \Phi}{\partial x_2} = i \int_0^S p_n dS + A, \tag{46}
\]

where \( A \) is a (complex) constant. Since \( p_n \) is a known function we can calculate

\[
i \int_0^S p_n dS = f(z) = f_1(z) + if_2(z), \tag{47}
\]

so that (40) and (47) lead to

\[
f(z) + A = f_1(z) + if_2(z) + A = \varphi(z) + z\varphi'(z) + \psi(z), \tag{48}
\]

where we used (see (39))

\[
\chi'(z) = \psi(z). \tag{49}
\]

Therefore the solution of the first fundamental boundary value problem is reduced to: given \( f(z) \) find two analytic functions \( \varphi(z) \) and \( \psi(z) \) that satisfy (48).

The important question concerning the results presented so far regards the uniqueness of the solutions \( \varphi(z) \) and \( \psi(z) \) of equations (42) or (48). This question is discussed for finite and infinite, simply and multiply connected regions in Muskhelishvilli (1966), Kalandiya (1975), and Parton and Perlin (1984), for example.

We present the solutions of (42) and (48) for the case of a circular region and for the case of a half plane. The solution for a circular region is especially important since by conformal mapping the given physical region may be transformed to a unit circle.

i) Assume that \( B \) is the unit circle

\[
|z| \leq 1. \tag{50}
\]

In the first fundamental boundary value problem we ask for the functions \( \varphi(z) \) and \( \psi(z) \), analytic in \( |z| < 1 \) and satisfying

\[
\varphi(z) + z\varphi'(z) + \overline{\psi(z)} = f(z), \tag{51}
\]

on \( |z| = 1 \). In (51) we set the constant \( A = 0 \). With \( f(z) \) given, the solution of (51) may be expressed in the form (see Lavrenteev and Shabat (1987, p. 327))

\[
\varphi(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)d\zeta}{\zeta - z} - \frac{z}{4\pi i} \int_C \frac{f(\zeta)}{\zeta^2} d\zeta; \tag{52}
\]

\[
\psi(z) = \frac{1}{2\pi i} \int_C \frac{\overline{f(\zeta)}d\zeta}{\zeta - z} + \frac{1}{4\pi iz} \int_C \frac{f(\zeta)}{\zeta^2} d\zeta - \frac{\varphi'(z)}{z},
\]

\]
where $C$ is the unit circle.

In the case of the second boundary value problem we write (42) in the form

$$K^* \phi(z) - z \overline{\phi'(z)} - \psi(z) = 2 \mu g(z),$$  \hspace{1cm} (53)

where $g(z) = \hat{u}_1 + i \hat{u}_2$. The functions $\phi(z)$ and $\psi(z)$, analytic in $|z| < 1$ and satisfying (53) for $|z| = 1$ could be expressed as

$$\phi(z) = \frac{\mu}{\kappa^* \pi i} \int_C \frac{g(\zeta) d\zeta}{\zeta - z} + \frac{c_1}{\kappa^* z}, \psi(z) = -\frac{\mu}{\pi i} \int_C \frac{g(\zeta) d\zeta}{\zeta - z} - \frac{\phi'(z)}{z} + \frac{c_1}{z}$$  \hspace{1cm} (54)

where

$$c_1 = \frac{\mu}{\pi i [(\kappa^*)^2 - 1]} \left\{ \kappa^* \int_C \frac{g(\zeta) d\zeta}{\zeta^2} + \int_C \frac{g(\zeta) d\zeta}{\zeta} \right\}.$$  \hspace{1cm} (55)

ii) Suppose that $B$ is the half plane

$$x_2 \leq 0.$$  \hspace{1cm} (56)

In the case of the first fundamental boundary value problem we have

$$\phi'(z) = \frac{1}{2 \pi i} \int_{-\infty}^{\infty} \frac{P(t) + iT(t)}{t - z} dt,$$  \hspace{1cm} (57)

where

$$P(t) = -\sigma_{22}(x_1 = t, x_2 = 0), \hspace{0.5cm} T(t) = \sigma_{12}(x_1 = t, x_2 = 0).$$  \hspace{1cm} (58)

With $\phi$ determined from (57) the function $\psi$ is obtained from

$$\psi'(z) = -\phi'(z) - \overline{\phi'(\bar{z})} - z \phi''(z).$$  \hspace{1cm} (59)

In the present case certain restrictions must be imposed on $P(t)$ and $T(t)$ so that the expression (57) satisfies

$$\phi'(z) = -\frac{X + iY}{2 \pi z} \frac{1}{z} + o\left(\frac{1}{z}\right),$$  \hspace{1cm} (60)

where $o(1/z)$ denotes terms that satisfy $\lim_{z \to \infty} [o(1/z)/(1/z)] = 0$. The stresses are determined from the function $\phi$ as

$$\sigma_{22} - i \sigma_{12} = \phi'(z) - \phi'(\bar{z}) + (z - \bar{z}) \phi''(z).$$  \hspace{1cm} (61)

In the case of the second boundary value problem first it is shown that the complex displacement could be written in the form

$$2\mu(\hat{u}_1 + i \hat{u}_2) = \kappa^* \phi(z) + \phi(\bar{z}) - (z - \bar{z}) \phi'(\bar{z}) + c,$$  \hspace{1cm} (62)
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where \( c \) is a constant. Then, we determine a function \( \chi \) as

\[
\chi(z) = \frac{\mu}{\pi i} \int_{-\infty}^{\infty} \frac{g_1(t) + ig_2(t)}{t - z} dt,
\]

where

\[
g_1(x) = \frac{d\bar{u}_1}{dx_1}, \quad g_2(x) = \frac{d\bar{u}_2}{dx_1}.
\]

With \( \chi(z) \) given by (63) the function \( \varphi'(z) \) is determined as (Lavrenteev and Shabat 1987, p. 332)

\[
\varphi'(z) = \begin{cases} 
\chi(z), & \text{if } \text{Im} \ z > 0 \\
-\frac{1}{\kappa^*} \chi(z), & \text{if } \text{Im} \ z < 0
\end{cases}
\]

(65)

The conformal mapping method is often used in connection with the complex variable method. Namely, we use an analytic function

\[
z = \omega(\zeta)
\]

(66)

to map a finite or infinite simply connected region in the plane of variable \( z \), bounded by a simple contour \( C \), in a one-to-one manner onto the unit circle \( |\zeta| < 1 \) in the \( \zeta \) plane. The existence of a function \( \omega \) is guaranteed by the Riemann mapping theorem. The fact that \( \omega(\zeta) \) is analytic implies that the mapping (66) is conformal; that is, the angles both in magnitude and sense, between the lines in the \( z \) plane and corresponding lines in the \( \zeta \) plane are equal. If the region in the \( z \) plane is finite then \( \omega(0) = 0 \). For the infinite regions in the \( z \) plane we have \( \omega(0) = \infty \).

Let \( r, \theta \) be cylindrical coordinates in the \( \zeta \) plane. If \( \mathbf{a} \) is an arbitrary vector defined at the point \( z \) then in the coordinate system with axes \( \bar{x}_1 - \bar{x}_2 \) it has components \( a_1, a_2 \). In the \( \zeta \) plane the coordinate lines \( r = \text{const.} \) and \( \theta = \text{const.} \) are mutually orthogonal. Let the unit vectors along the coordinate lines \( r = \text{const.} \) and \( \theta = \text{const.} \) be denoted by \( \mathbf{e}_r \) and \( \mathbf{e}_\theta \), respectively. The vector \( \mathbf{a} \) can be represented as

\[
\mathbf{a} = a_r \mathbf{e}_r + a_\theta \mathbf{e}_\theta.
\]

(67)

The components \( a_1, a_2, a_r, \) and \( a_\theta \) are connected as

\[
a_r + ia_\theta = (a_1 \cos \alpha + a_2 \sin \alpha) + i(-a_1 \sin \alpha + a_2 \cos \alpha),
\]

(68)

or

\[
a_r + ia_\theta = \exp(-i\alpha)(a_1 + ia_2),
\]

(69)

where \( \alpha \) is the angle between the unit vector \( \mathbf{e}_1 \) and the unit vector \( \mathbf{e}_r \) with the direction from \( \mathbf{e}_1 \) to \( \mathbf{e}_r \) taken as positive. We determine next the angle
To this end we consider the change of $z$ in the $e_r$ direction. Then $z$ becomes $z + dz$ with $dz$ given as

$$dz = |dz| \exp(i\alpha). \quad (70)$$

Since the point $\zeta = r \exp(i\theta)$ that corresponds to $z$ is displaced in the radial ($e_r$) direction we have

$$d\zeta = |d\zeta| \exp(i\theta). \quad (71)$$

However from (66) we have $dz = \omega'(\zeta) d\zeta$, so that

$$\exp(i\alpha) = \frac{dz}{|dz|} = \exp(i\theta) \frac{\omega'(\zeta)}{\omega'(\zeta)}.$$ 

From (72) it follows that

$$\exp(-i\alpha) = \frac{\zeta}{r} \frac{\omega'(\zeta)}{|\omega'(\zeta)|}. \quad (73)$$

With (73) equation (69), when applied to displacement vector $u$, gives

$$u_r + iu_\theta = \frac{\zeta}{r} \frac{\omega'(\zeta)}{|\omega'(\zeta)|} (u_1 + iu_2). \quad (74)$$

For the components of the stress tensor it could be easily shown that

$$\sigma_{rr} + \sigma_{\theta\theta} = \sigma_{11} + \sigma_{22};$$

$$\sigma_{\theta\theta} - \sigma_{rr} + 2i\sigma_{\theta r} = \exp(2i\alpha)(\sigma_{22} - \sigma_{11} + 2i\sigma_{12}). \quad (75)$$

From (72) we have

$$\exp(2i\alpha) = \frac{\zeta^2}{r^2} \frac{(\omega'(\zeta))^2}{|\omega'(\zeta)|^2} = \frac{\zeta^2 \omega'(\zeta)}{r^2 \omega'(\zeta)}. \quad (76)$$

Let for any analytic $\varphi$ and $\psi$,

$$\varphi(z) = \varphi(\omega(\zeta)) = \varphi_1(\zeta); \quad \psi(z) = \psi(\omega(\zeta)) = \psi_1(\zeta). \quad (77)$$

Then from (75),(76),(41),(66), and (77) we obtain

$$\sigma_{rr} + \sigma_{\theta\theta} = 2[\Phi_1(\zeta) + \Phi_1(\zeta)];$$

$$\sigma_{\theta\theta} - \sigma_{rr} + 2i\sigma_{\theta r} = \frac{2\zeta^2}{r^2 \omega'(\zeta)}[\omega(\zeta)\Phi'_1(\zeta) + \omega'(\zeta)\Psi_1(\zeta)], \quad (78)$$

where $\Phi_1(\zeta) = \varphi'(z(\zeta)); \quad \Psi_1(\zeta) = \psi'(z(\zeta))$. Also, the boundary conditions for the first and second boundary value problem (48) and (42) become

$$f_1(z(\zeta)) + if_2(z(\zeta)) + A = \varphi_1(\zeta) + \frac{\omega(\zeta)}{\omega'(\zeta)}\varphi'_1(\zeta) + \psi_1(\zeta);$$
6.4 Complex variable method for plane problems

\[ 2\mu(\hat{u}_1 + i\hat{u}_2) = \left[ \kappa^* \varphi_1(\zeta) - \frac{\omega(\zeta)}{\omega'(\zeta)} \varphi'_1(\zeta) - \psi_1(\zeta) \right]. \tag{79} \]

Equations (79) must be satisfied on the boundary \(|\zeta| = 1\) in the \(\zeta\) plane.

We now present some concrete problems and their solutions obtained by the complex variable method.

i) Suppose that the functions \(\varphi\) and \(\psi\) have the form

\[ \varphi(z) = C\imath^2 z^2; \quad \psi(z) = -C\imath z^2. \tag{80} \]

By using (26), (30) we obtain

\[ \sigma_{11} + \sigma_{22} = 2[2C\imath z - 2C\imath \bar{z}] = -8Cz; \]
\[ \sigma_{22} - \sigma_{11} + 2i\sigma_{12} = 2[2\bar{z}C\imath - 2C\imath z] = 4C(\bar{z} - z) = 8Cz. \tag{81} \]

From (80), (81) it follows that

\[ \sigma_{12} = 0; \quad \sigma_{11} = -8Cz; \quad \sigma_{22} = 0. \tag{82} \]

The stress distribution (82) corresponds to pure bending. By using (19) we get

\[ u_1 = -\frac{8C}{E}(1 - \nu^2)x_1x_2; \quad u_2 = \frac{4C}{E}(1 + \nu)[(1 - \nu)x_1^2 + \nu x_2^2]. \tag{83} \]

In the elementary bending theory by constant bending moment \(M_0\) we have

\[ \sigma_{11} = \frac{M_0}{J}x_2, \tag{84} \]

where \(J\) is the axial moment of inertia. By comparing (84) and (82) we conclude that \(8c = -M_0/J\). With this value (83) becomes

\[ u_2 = -\frac{M_0}{EJ}(1 - \nu^2) \left[ \frac{x_1^2}{2} + \frac{\nu}{1 - \nu} \frac{x_2^2}{2} \right]. \tag{85} \]

Note that in the limit when \(\nu \to 0\), we obtain from (85)

\[ EJ\frac{d^2u_2}{dx_1^2} = -M_0. \tag{86} \]

ii) Consider now the problem ii) of Section 6.3 shown in Fig. 4.

Then, we have

\[ \sigma_{22}(x_2) = -p; \quad -a \leq x_2 \leq a, \]
\[ \sigma_{12} = 0; \quad -a \leq x_2 \leq a. \tag{87} \]
Chapter 6 Plane State of Strain and Plane State of Stress

From (57), (59) it follows that

\[ \varphi'(z) = -\frac{p}{2\pi i} \int_{-\pi}^{\pi} \frac{dt}{t-z} = \frac{p}{2\pi i} \ln \frac{z-a}{z+a}; \]

\[ \psi'(z) = -\frac{zp}{2\pi i} \int_{-\pi}^{\pi} \frac{dt}{(t-z)^2} = -\frac{paz}{\pi i(z^2-a^2)}. \]  

By using (88) in (41) we obtain

\[ \sigma_{11} + \sigma_{22} = 2[\varphi'(z) + \bar{\varphi}'(\bar{z})] = 4\Re(\varphi'(z)) = -\frac{2p}{\pi} (\theta_2 - \theta_1); \]

\[ \sigma_{22} - \sigma_{11} + 2i\sigma_{12} = 2[\bar{z}\varphi''(z) + \psi'(z)] = \frac{2pa}{\pi i} \frac{\bar{z} - z}{z^2 - a^2}, \]  

where \( \theta_1 \) and \( \theta_2 \) are defined as in Fig. 4. From (89) it follows that the stress field has the same value as was determined earlier.

iii) An infinite plate with an elliptical hole.

Consider a plate with an elliptical hole as shown in Fig. 8. Suppose in the \( z \) plane the ellipse is defined as

\[ \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1, \]  

where \( a \) and \( b \) are half axes of the ellipse. The function

\[ z(\zeta) = \omega(\zeta) = A \left( \frac{1}{\zeta} + c\zeta \right), \]  

where

\[ A = \frac{a+b}{2}; \quad c = \frac{a-b}{a+b}, \]  

maps the \( z \) plane without ellipse onto a unit circle in the \( \zeta \) plane (Fig. 8).

![Figure 8](image-url)
Suppose that the plate is loaded so that at the points with large distance from the hole the stress field is uniaxial
\[ \sigma_{11} = 0; \quad \sigma_{22} = \sigma; \quad \sigma_{12} = 0, \] (93)
and that the hole is stress free. Consider the functions
\[ \varphi_1(\zeta) = \frac{\sigma A}{4} \left( \frac{1}{\zeta} - (1 + c)\zeta \right); \quad \psi_1(\zeta) = \frac{\sigma A}{2} \left[ \frac{1}{\zeta} - \frac{(1 + c + c^2)\zeta + \zeta^3}{1 - c\zeta^2} \right]. \] (94)
By using
\[ \Phi_1(\zeta) = \frac{\varphi'(\zeta)}{\omega'(\zeta)} = \frac{\sigma}{4} \frac{1 + (2 + c)\zeta^2}{1 - \zeta^2}, \] (95)
in the expressions (78) we obtain
\[ \sigma_{\theta\theta} = 2[\Phi_1(\zeta) + \Phi_1(\zeta)]; \quad \sigma_{rr} = 0, \] (96)
on \[ |\zeta| = 1, \] that is, on \[ \zeta = \exp(i\theta), \theta \in [0, 2\pi]. \] By using \[ \zeta = \exp(i\theta) \] in (96) it follows that
\[ \sigma_{\theta\theta} = \sigma \frac{1 - (2 + c)c + 2\cos(2\theta)}{1 + c^2 - 2c\cos(2\theta)}. \] (97)
The maximal value of \( \sigma_{\theta\theta} \) is given as
\[ (\sigma_{\theta\theta})_{\text{max}} = \sigma \left( 1 + \frac{2 + c}{1 - c} \right) = \sigma \left( 1 + \frac{2a}{b} \right), \] (98)
for \( \theta = 0 \) and \( \theta = \pi. \) The minimal value of \( \sigma_{\theta\theta} \) is
\[ (\sigma_{\theta\theta})_{\text{min}} = -\sigma, \] (99)
for \( \theta = \pi/2 \) and \( \theta = -\pi/2. \) The solution (97) was obtained by Kolossov. It could be used to study the basic problem of crack mechanics, the so-called Griffith crack problem (Griffith 1921). As the crack model we take an elliptical hole with the axis \( b \) small (see Fig. 9). The function (91) in this case becomes \( c = 1, = A = a/2, \)
\[ z(\zeta) = \omega(\zeta) = \frac{a}{2} \left( \frac{1}{\zeta} + \zeta \right). \] (100)
With (100) the functions (94) are
\[ \varphi_1(\zeta) = \frac{\sigma a}{8} \left( \frac{1}{\zeta} - 3\zeta \right); \quad \psi_1(\zeta) = \frac{\sigma a}{4} \frac{1 - 4\zeta^2 - \zeta^4}{\zeta(1 - \zeta^2)}. \] (101)
To obtain stresses in the \( z \) plane we solve (100) for \( \zeta \) to obtain
\[ \zeta = \frac{z}{a} - \sqrt{\left( \frac{z}{a} \right)^2 - 1}. \] (102)
By using (102) in (101) it follows that

\[
\varphi(z) = \frac{\sigma a}{4} \left[ 2 \sqrt{\left(\frac{z}{a}\right)^2 - 1} - \frac{z}{a} \right], \quad \psi(z) = -\frac{\sigma a}{2} \left[ \sqrt{\left(\frac{z}{a}\right)^2 - 1} - \frac{z}{a} \right].
\]

(103)

With (103) and (26),(30) the stress tensor components can be determined. Along the \(x_1\) axis (i.e., \(x_2 = 0; z = \bar{z} = x_1\)) we have

\[
\sigma_{11} = \begin{cases} 
\sigma \left[ \frac{x_1}{a} \sqrt{\left(\frac{x_1}{a}\right)^2 - 1} - 1 \right], & x_1 > a \\
-\sigma, & x_1 < a
\end{cases}
\]

\[
\sigma_{22} = \begin{cases} 
\sigma \sqrt{\left(\frac{x_1}{a}\right)^2 - 1}, & x_1 > a \\
0, & x_1 < a
\end{cases}
\]

(104)

Note that the stresses (104) become infinite at the tip of the crack. This is a consequence of undefined curvature at that point. There are crack models where this singularity is avoided.
Problems 2&3

To determine the displacement field corresponding to (103) we use equation (20)

\[ 2\mu(u_1 + iu_2) = \kappa^* \varphi(z) - z\varphi'(z) - \psi(z). \]  

(105)

Since the surface of the crack is stress free \((p_n = 0 \text{ in } (47))\) we have from equation (48),

\[ \varphi(z) + z\varphi'(z) + \psi(z) = 0, \]  

(106)
on \(x_1 = 0, -a < x_1 < a.\) By evaluating (105) on the crack boundary and by combining the result with (106) we get

\[ 2\mu(u_1 + iu_2) = (\kappa^* + 1)\varphi(z); \quad \text{on } -a < x_1 < a. \]  

(107)

It is of interest to determine the component of the displacement vector in the \(\bar{x}_2\) direction. Thus, from (107) it follows that

\[ 2\mu u_2 = (\kappa^* + 1)\Im[\varphi(z)]. \]  

(108)

Where \(\Im[\varphi(z)]\) is the imaginary part of \(\varphi(z)\). For the Griffith crack we obtain the displacement when (103) is used in (108)

\[ u_2 = \frac{1 + \kappa^*}{4\mu} \sqrt{a^2 - x_1^2}. \]  

(109)

From (109) it follows that the displacement component \(u_2\) at the tip of the crack is equal to zero.

Problems

1. Show that Hooke's law solved for the components of the strain tensor could be written in the Cartesian and cylindrical coordinate system

\[ E_{11} = \frac{1}{E} (\sigma_{11} - \nu \sigma_{22}); \quad E_{22} = \frac{1}{E} (\sigma_{22} - \nu \sigma_{11}); \quad E_{12} = \frac{\sigma_{12}}{2\mu}; \]

\[ E_{rr} = \frac{1}{E} (\sigma_{rr} - \nu \sigma_{\theta\theta}); \quad E_{\theta\theta} = \frac{1}{E} (\sigma_{\theta\theta} - \nu \sigma_{rr}); \quad E_{r\theta} = \frac{1 + \nu}{E} \sigma_{r\theta}, \]  

(a)

for the plane stress case. For the plane strain case it is necessary to replace \(E\) and \(\nu\) with \(E_1\) and \(\nu_1\) given by

\[ E_1 = \frac{E}{1 - \nu^2}; \quad \nu_1 = \frac{\nu}{1 - \nu}. \]  

(b)

2. Consider an infinite wedge shown in the following figure. The wedge is loaded by a concentrated force \(F\). Show that the function

\[ \Phi = f[(A + B\theta) \cos \theta + (C + D\theta) \sin \theta], \]

where \(A, B, C,\) and \(D\) are constants, is a solution of the biharmonic equation (6.5-3).
Determine the constants so that the sides of the wedge are stress free. Thus, show that

\[ A = 0; \quad C = 0; \quad B = 0; \quad D = -\frac{2F}{2\alpha + \sin 2\alpha}; \quad \sigma_{rr} = \frac{2D \cos \theta}{r}. \]

3. The thin cantilever shown in the following figure is loaded on the upper side with shear \( \sigma_{12} = \tau_0 = \text{const.} \) Show that the function

\[ \Phi = \frac{\tau_0}{4} \left( x_1 x_2 - \frac{x_1 x_2^2}{h} - \frac{x_1 x_2^3}{h^2} + \frac{L x_2^2}{h} + \frac{L x_2^3}{h^2} \right), \]

satisfies equation (6.2-13). Are all boundary conditions satisfied?

4. A thin rectangular elastic plate with sides \( a \) and \( b \) rotates with constant angular velocity \( \omega \) about an axis that coincides with the side of length \( b \). Show that the Airy stress function is a solution of the following equation

\[ \Delta \Delta \Phi = (1 - \nu) \rho_0 \omega^2, \]

where \( \rho_0 \) is the density of the plate.

5. Show that the function

\[ \Phi = C_1 \theta + C_2 + C_3 \cos 2\theta + C_4 \sin 2\theta, \]
where $C_i$, $i = 1, 2, 3, 4$ are constants satisfies the biharmonic equation in polar coordinates (6.2-24).

6. Starting with equations (6.3-55) and using Hooke’s law (6.1-4) show that the components of the displacement vector for the half plane loaded by a concentrated force are (see Love (1944, p.211) )

$$u_1 = -\frac{F(\lambda + 2\mu)}{2\pi(\lambda + \mu)} \ln r - \frac{F}{2\pi\mu} \frac{x_2^2}{r^2};$$

$$u_2 = -\frac{F}{2\pi(\lambda + \mu)}(\theta_1 - \frac{\pi}{2}) + \frac{F}{2\pi\mu} \frac{x_1x_2}{r^2}.$$

where $r = (x_1^2 + x_2^2)^{1/2}$ and $\theta_1$ is the angle between the position vector of the point with coordinates $(x_1, x_2)$ and the $x_1$ axis (see Fig. 4).

7. A thin circular disc of the radius $R$ rotates about an axis that passes through its center and is orthogonal to the plane of the disc, with the constant angular velocity $\omega$. By using the plane stress equations (special case of (1.5-6))

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial\sigma_{r\theta}}{\partial \theta} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) + f_r = 0;$$

$$\frac{\partial\sigma_{\theta r}}{\partial r} + \frac{1}{r} \frac{\partial\sigma_{\theta\theta}}{\partial \theta} + \frac{2}{r} \sigma_{\theta r} + f_\theta = 0,$$

and the results of Problem 1, show that the radial stress tensor components

$$\sigma_{rr} = \frac{3 + \nu}{8} \rho_0 \omega^2 R^2 \left[1 - \left(\frac{r}{R}\right)^2\right];$$

$$\sigma_{\theta\theta} = \frac{3 + \nu}{8} \rho_0 \omega^2 R^2 \left[1 - \frac{1}{3 + \nu} \left(\frac{r}{R}\right)^2\right],$$

and that the maximal values of stresses is at $r = 0$,

$$\sigma_{rr} = \sigma_{\theta\theta} = \frac{3 + \mu}{8} \rho_0 \omega^2 R^2.$$

8. Show that (6.3-31) after the use of (6.3-30) leads to the following expression for stresses

$$\sigma_{11} = \frac{q}{2I} x_1^2 x_2 - \frac{q}{2I} \left(\frac{2}{3} x_2^3 - \frac{h^2}{10} x_2\right);$$

$$\sigma_{22} = \frac{q}{2I} \left(\frac{1}{3} x_1^3 - \frac{h^2}{4} x_2 - \frac{h^3}{12}\right);$$

$$\sigma_{12} = \frac{q}{2I} \left(\frac{h^2}{4} - x_2^2\right) x_1,$$

where $I = \delta h^3/12$. 
9. Consider the half plane shown in Fig. 3. Suppose that the plane is loaded by concentrated force \( \mathbf{F} = F \mathbf{e}_2 \) at the point \((0, 0)\). Following the same procedure as in Problem ii) of Section 6.3 show that

\[
\begin{align*}
\sigma_{11} &= -\frac{2F}{\pi} \frac{x_2^2}{(x_2^2 + x_1^2)^2}; \\
\sigma_{22} &= -\frac{2F}{\pi} \frac{x_2^3}{(x_2^2 + x_1^2)^2}; \\
\sigma_{12} &= -\frac{2F}{\pi} \frac{x_1 x_2}{(x_1^2 + x_2^2)^2}.
\end{align*}
\]

10. Consider a wedge, shown in the figure that is loaded by a concentrated couple acting in a plane of the wedge at its tip. Show that

\[
\Phi = A\theta + D \sin 2\theta,
\]

where \(A\) and \(D\) are constants is a solution of equation (6.2-24). Furthermore, by using (6.2-25) determine the components of the stress tensor

\[
\begin{align*}
\sigma_{rr} &= -\frac{4D}{r^2} \sin 2\theta; \quad \sigma_{\theta\theta} = 0; \\
\sigma_{r\theta} &= \frac{A}{r^2} + \frac{2D}{r^2} \cos 2\theta.
\end{align*}
\]

By applying the boundary condition

\[
\sigma_{r\theta}(r, \theta = \pm \alpha) = 0,
\]

show that

\[
\begin{align*}
\sigma_{rr} &= \frac{2M \sin 2\theta}{r^2(2\alpha - \tan 2\alpha) \cos 2\alpha}; \quad \sigma_{\theta\theta} = 0; \\
\sigma_{r\theta} &= \frac{M}{r^2(2\alpha - \tan 2\alpha)} \left(1 - \frac{\cos 2\theta}{\cos 2\alpha}\right).
\end{align*}
\]
Chapter 7

Energy Method in Elasticity Theory

7.1 Introduction

In general in physics and particularly in mechanics the concepts of work and energy play an important role that can be traced even in the works of Aristotle. Work and energy are invariant quantities since they do not depend on the choice of coordinate system. In addition, many fundamental laws of mechanics can be expressed in two ways: first by the requirement that certain differential equations are satisfied (for example, in the equilibrium state of an elastic body the differential equations (1.3-10) hold) or by the requirement that a specific quantity in a given state (motion or equilibrium) is in minimum. The second approach comprises what is called the variational methods of mechanics. In this method work and energy are of fundamental importance.

In the variational method two types of variational principles are formulated: integral and differential. In both cases they express the mathematical requirement that a certain quantity is during the process (motion, equilibrium) minimal or maximal (sometimes only stationary).

There exists a belief, stated by Leibniz, that all processes in nature (and therefore mechanical processes as a special case) are, in a certain sense, optimal.¹ This belief is present in all branches of physics.²

When a variational formulation for a given problem is known, then it can be used in several different directions (see, for example, Vujanovic and Jones (1989)).

1. The differential equations of the process together with corresponding boundary conditions could be derived.


2. The variational principle could be used for the study of conservation laws (finding of the first integrals of the differential equations describing the process) by use of the Noether theorem.

3. Approximate solutions of the differential equations of the process (Ritz method) could be obtained.

Therefore there is permanent interest in formulating variational principles for the problems of elasticity theory.

7.2 Work and inner energy

The first law of thermodynamics states that the change of the total energy of a thermoelastic body is equal to the work of outer forces plus the change of amount of heat. In what follows we consider elastic bodies only. Thus there is no change of heat during any deformation process. We begin with the problem of determining work of outer forces.

Consider an elastic body B shown in Fig. 1. Suppose that the body is loaded by a system of forces $Q_i, i = 1, 2, \ldots, n$. We assume that the $Q_i$ are so-called generalized forces. By generalized forces we mean forces and couples.

At the point of application of the force $Q_i$ we define a generalized displacement $q_i$. It represents the displacement of the point of application of the force $Q_i$ in the direction of the force. In the case when $Q_i$ is a couple, $q_i$ represents the angle of rotation about the axis of the couple. Before we determine the work for the case shown in Fig. 1 we consider a body loaded by a single generalized force $Q$ (see Fig. 2).

The problem of determining work of surface forces for the general case shown in Fig. 1 is treated in Section 7.4. When the single force $Q$ is applied the body deforms from configuration $\kappa_0$ to configuration $\kappa$. Suppose that the generalized displacement corresponding to $Q$ is $q$. Suppose further that the dependence of $Q = |Q|$ on $q$ is as shown in Fig. 3.
Then, if $q$ is increased to $q + dq$, the work done by the force $Q$ is given as

$$dW = Qdq.$$  

(1)

The change of total mechanical energy (potential and kinetic) of the body is equal to the work of outer forces and the change of the amount of thermal energy (heat) of the body. If the change of the thermal energy is equal to zero we have

$$dE_k + dU = dW,$$  

(2)

where $dE_k$ is the change of kinetic energy, $dU$ is the change of potential energy, and $dW$ is the work of the outer forces. When the change of kinetic energy is equal to zero we have

$$dU = dW.$$  

(3)

The total potential energy of the body, also called the strain energy, is then

$$U = \int_0^q Q(\lambda)d\lambda,$$  

(4)

where we assumed that the initial state $q = 0$, we have $U = 0$.

Next we define the complementary energy by the relation

$$dU_c = qdQ = dW_c,$$  

(5)

3The quantity $dW_c = qdQ$ is called the pseudowork.
so that

$$U_c = \int_0^Q q(\lambda) d\lambda .$$

(6)

In Fig. 4 we show the geometrical meaning of $U$ and $U_c$. From Fig. 4 we conclude that

$$U_c = Qq - U .$$

(7)

The change from $U$ to $U_c$ may be viewed as the Legendre transformation (sometimes called the Friedrich transformation). It plays an important role in mechanics (see, e.g., Pal and Anisiu (1995)).

![Figure 4](image)

We turn now to the problem of determining $dU$ for the linearly elastic body that is in a three-dimensional stress state. We write

$$dU = \int_V u dV ,$$

(8)

where $u$ is to be determined.

Consider a body $B$ in a deformed state. Let $u$ be the displacement vector of an arbitrary point and let $\sigma_{ij}$ be the components of the stress tensor at this point expressed in a fixed Cartesian coordinate system $\bar{x}_i$. We consider an elementary parallelepiped with the sides $dx_i$, $i = 1, 2, 3$. If we deform the body additionally the displacement vector in the new equilibrium position will be $u + \bar{u}$. We assume that $\bar{u}$ is small. The strain tensor will change from $E_{ij}$ in the original to $\bar{E}_{ij} + \tilde{E}_{ij}$ in the new configuration. The stress tensor components $\sigma_{ij} = \sigma_{ij}(E_{ij})$ will change also so that in the new equilibrium configuration we have $\sigma_{ij} + \tilde{\sigma}_{ij}$. By using the fact that $\bar{u}$ is small we obtain

$$\tilde{E}_{ij} = \frac{1}{2} \left( \frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_i}{\partial x_i} \right) ; \quad \tilde{\sigma}_{ij} = \frac{\partial \sigma_{ij}}{\partial E_{mk}} \bar{E}_{mk} .$$

(9)

In Fig. 5a we show the elementary parallelepiped and in Fig. 5b the dependence of one component of the stress tensor ($\sigma_{22}$) on one component of the strain tensor ($E_{22}$). Note that this relation is not assumed to be linear. Thus the result that follows will be valid for materially nonlinear elastic materials.
Next we determine the work of forces acting on the sides of the parallelepiped. On side I a force of the intensity \( |\sigma_{22} + (\partial \sigma_{22}/\partial x_2)dx_2| dx_1 dx_3 \) acts. The displacement of its point of application is \( |\bar{u}_2 + (\partial \bar{u}_2/\partial x_2)dx_2| dx_2 \). Therefore work is

\[
A_1 = \left\{ \left( \sigma_{22} + \frac{\partial \sigma_{22}}{\partial x_2} dx_2 \right) dx_1 dx_3 \right\} \left( \bar{u}_2 + \frac{\partial \bar{u}_2}{\partial x_2} dx_2 \right).
\] (10)

On side II the force is \( \sigma_{22} dx_1 dx_3 \) and the displacement is \( \bar{u}_2 \) so that the work is

\[
A_2 = -\sigma_{22} \bar{u}_2 dx_1 dx_3.
\] (11)

The total work of \( \sigma_{22} \) on sides I and II becomes

\[
\frac{\partial}{\partial x_2} (\sigma_{22} \bar{u}_2) dx_1 dx_2 dx_3.
\] (12)

By using the same procedure we can determine the work of other components of the stress vector (i.e., \( \sigma_{21} \) and \( \sigma_{23} \)) on the sides I and II. Thus the total work of forces acting on sides I and II becomes

\[
\frac{\partial}{\partial x_2} (\sigma_{22} \bar{u}_2 + \sigma_{12} \bar{u}_1 + \sigma_{23} \bar{u}_3) dx_1 dx_2 dx_3.
\] (13)

The work of the stress vector on the other two pairs of sides should be added to (13) so that the work of all surface forces acting on the sides of the parallelepiped is

\[
A_s = \frac{\partial}{\partial x_i} (\sigma_{ij} \bar{u}_j) dx_1 dx_2 dx_3.
\] (14)

The work of body forces is simply

\[
A_b = (f_i \bar{u}_i) dx_1 dx_2 dx_3.
\] (15)

so that the total work of all surface and body forces is

\[
\sum \sigma_{ij} dE_{ij} + \bar{u}_i \left( \frac{\partial \sigma_{ij}}{\partial x_j} + f_i \right) dx_1 dx_2 dx_3,
\] (16)
where \( dE_{ij} = \ddot{E}_{ij} \) and is given by (9). By using the equilibrium equations and (3) we conclude that

\[
dU = \int_V \int \sigma_{ij} dE_{ij} dV.
\]

(17)

Therefore \( u \) in (8) becomes \( u = \sigma_{ij} dE_{ij} \).

If we want to calculate \( U \) between two configurations with the strain tensor \( E^0_{ij} \) and \( E^1_{ij} \) we have to integrate (17). In general this integration will be path dependent. However, if \( \sigma_{ij} \), as in our case, depends on \( E_{ij} \) only and the integration is path independent, the function \( U \) can be defined as\(^4\)

\[
U = \int_V \int \left( \int_{E^0_{km}}^{E^1_{km}} \sigma_{ij}(E_{km}) dE_{ij} \right) dV.
\]

(18)

By using similar arguments we can obtain the expression for the total complementary work in the form

\[
dU_c = \int_V \int E_{ij} d\sigma_{ij} dV,
\]

(19)

or

\[
U_c = \int_V \int \left( \int_{\sigma^0_{km}}^{\sigma^1_{km}} E_{ij}(\sigma_{km}) d\sigma_{ij} \right) dV,
\]

(20)

where \( \sigma^0_{ij} \) and \( \sigma^1_{ij} \) are components of the stress tensor in the initial and final configuration.

We turn now to the linearly elastic body. Suppose that the body deforms between the states \( E^0_{ij} \) and \( E^1_{ij} \). Let \( t \in [0,1] \) be a real parameter and let \( E^t_{ij} \) be defined as

\[
E^t_{ij} = E^0_{ij} + t( E_{ij} - E^0_{ij} ).
\]

(21)

For linearly elastic materials we have \( \sigma_{ij} = C_{ijkl} E_{kl} \) so that by using this and (21) in (18) we obtain

\[
U = \frac{1}{2} \int_V \int \sigma_{ij} E_{ij} dV,
\]

(22)

where we assumed that \( E^0_{ij} = 0 \). The function

\[
u = \frac{1}{2} \sigma_{ij} E_{ij},
\]

(23)

is called the \textit{strain energy density} or \textit{elastic potential}. The elastic potential could be expressed in two equivalent ways. If we use Hooke's law in the form (3.4-14) we get

\[
u = \frac{1}{2E} [(1 + \nu)\sigma_{mn}\sigma_{mn} - \nu\Theta^2],
\]

(24)

\(^4\)As a matter of fact, the existence of \( U \) characterizes the so-called Green's elasticity. The elasticity theory defined by (3.1-5) is sometimes called the Cauchy elasticity.
or
\[ u = \frac{1}{2E} \left[ \sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2 - 2\nu(\sigma_{11}\sigma_{22} + \sigma_{11}\sigma_{33} + \sigma_{22}\sigma_{33}) \right] + 2(1 + \nu)(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{13}^2). \]  
(25)

Similarly if we use Hooke’s law
\[ \sigma_{ij} = \lambda \delta_{ij} + 2\mu E_{ij}, \]  
(26)
in (23) we obtain
\[ u = \frac{1}{2} \lambda (E_{nn})^2 + \mu E_{ij} E_{ij}, \]  
(27)
or
\[ u = \frac{1}{2} \lambda (E_{11} + E_{22} + E_{33})^2 + \mu (E_{11}^2 + E_{22}^2 + E_{33}^2) + 2\mu (E_{12}^2 + E_{13}^2 + E_{23}^2). \]  
(28)

From (28) it follows that \( u \) is a positive definite form in the strain tensor components \( E_{ij} \) since both \( \lambda \) and \( \mu \) are positive. The expression (28) was used in Section 4 (see (4.4-7)). Since (26) is a linear stress strain law it follows that (24) is also a positive definite quadratic form in the stress tensor components \( \sigma_{ij} \). We emphasize again that (18) is valid for nonlinear elastic materials, whereas (23) holds for linear materials only.

### 7.3 Betti’s theorem

We state now a general reciprocity law that connects two states of equilibrium on an elastic body that are realized by two different external loads.

Suppose that an elastic body \( \mathcal{B} \) with volume \( V \) and boundary surface \( S \) is in equilibrium in two configurations, I and II. Let
\[ \sigma_{ij}^I, \quad E_{ij}^I, \quad u_i^I, \]  
(1)
be stress, strain, and displacement fields due to the action of surface forces \( p_i^I \) and body forces \( f_i^I \). Similarly, let
\[ \sigma_{ij}^{II}, \quad E_{ij}^{II}, \quad u_i^{II}, \]  
(2)
be a stress, strain, and displacement fields corresponding to the surface and body forces \( p_i^{II} \) and \( f_i^{II} \). Since
\[ \sigma_{ij}^I = C_{ijkl} E_{kl}^I, \quad \sigma_{ij}^{II} = C_{ijkl} E_{kl}^{II}, \]  
(3)
we have
\[ \sigma_{ij}^I E_{ij}^I = C_{ijkl} E_{kl}^I E_{ij}^I = C_{kl} E_{kl}^I E_{ij}^I = \sigma_{ij}^{II} E_{ij}^I = \sigma_{ij}^{II} E_{ij}^I. \]  
(4)
where we used the symmetry property of the tensor of elasticity. From (4) it follows that

$$\int \int_V \int \sigma_{ij}^I E_{ij}^I dV = \int \int \int \sigma_{ij}^{II} E_{ij}^I dV. \tag{5}$$

The expression (5) is a special form of Betti’s theorem. We express it differently. To do this we transform the left-hand side of (5) as follows

$$\int \int_V \int \sigma_{ij}^I \frac{1}{2} \left( \frac{\partial u_i^{II}}{\partial x_j} + \frac{\partial u_j^{II}}{\partial x_i} \right) dV = \int \int \int \sigma_{ij}^I \frac{\partial u_i^{II}}{\partial x_j} dV$$

$$= \int \int_V \int \frac{\partial}{\partial x_j} (\sigma_{ij}^I u_i^{II}) dV - \int \int_V \int \frac{\partial \sigma_{ij}^I}{\partial x_j} u_i^{II} dV, \tag{6}$$

where we used the symmetry of the tensor $\sigma_{ij}$. When the Gauss theorem and the equilibrium equations (1.3-10) are used in (6), we obtain

$$\int \int \int \sigma_{ij}^I E_{ij}^I dV = \int \int \int p_i^{II} u_i dS + \int \int \int f_i^{II} u_i^{II} dV. \tag{7}$$

By using the same procedure to transform the right-hand side of (5) it follows that

$$\int \int \int p_i^I u_i^{II} dS + \int \int \int f_i^I u_i^{II} dV = \int \int \int p_i^{II} u_i^I dS + \int \int \int f_i^{II} u_i^I dV. \tag{8}$$

The relation (8) is Betti’s theorem: the work of outer forces $I$ on the displacement field corresponding to the system of forces $II$ is equal to the work of the system of forces $II$ on the displacement field corresponding to the system of forces $I$.

The importance of Betti’s theorem lies in the fact that the system of forces $I$ and $II$ can be chosen arbitrarily. For example, suppose that we are interested in the displacements corresponding to the system of forces $I$. We can choose a very simple form for the system of forces $II$ (for which we know the solution) and then obtain from (8) an integral representation of the unknown displacement field. The method is used in the boundary integral method for the solution of elasticity problems.

Betti’s theorem could be extended to dynamic problems of elasticity by adding the inertial force $-\rho_0 \partial^2 u_i/\partial t^2$ to the body forces $f_i$, so that

$$\int \int \int p_i^I u_i^{II} dS + \int \int \int \left( f_i^I - \frac{\partial^2 u_i^I}{\partial t^2} \right) u_i^{II} dV$$

$$= \int \int \int p_i^{II} u_i^I dS + \int \int \int \left( f_i^{II} - \frac{\partial^2 u_i^{II}}{\partial t^2} \right) u_i^I dV. \tag{9}$$
7.4 Maxwell's theorem

If the body is loaded by concentrated forces acting on the boundary $S$ and body forces are equal to zero, then (8) leads to

$$\sum_{i=1}^{k} Q_i^I q_i^I = \sum_{i=1}^{k} Q_i^{II} q_i^I,$$

(10)

where we used the properties of the Dirac function and the notation from Figures 1 and 2. The expression (9) has importance in structural mechanics.

7.4 Maxwell's theorem

For the case of a single concentrated force and linear dependence between the force and displacement of its point of application the work is determined from (7.2-1). Namely, since

$$W = \int_0^q Q(\lambda) d\lambda,$$

(1)

for the case of linear force displacement law

$$Q = \Delta q,$$

(2)

with $\Delta = \text{const}$. Equation (1) then gives

$$W = \frac{1}{2} \Delta q^2.$$

(3)

For the case of $N$ forces acting on the body (Fig.1) we have

$$dW = \sum_{i=1}^{N} Q_i dq_i.$$

(4)

Integration of (4) is not as easy as in the case of (1),(2) since each $q_i$ depends on all $Q_j$. We write this as $q_i = q_i(Q_1, \ldots, Q_N)$.

To determine $W$ from (4) the dependence of $Q_i$ on $q_1, q_2, \ldots, q_N$ is needed. Next we determine this dependence. Let $\Delta_{mn}$ be the displacement of a point $n$ caused by a unit (generalized) force acting at the point $m$. Then $\Delta_{nm}$ is a generalized displacement of a point $m$ caused by a unit force acting at $n$. From Betti's law (7.3-10) we obtain

$$\Delta_{nm} = \Delta_{mn},$$

(5)

The coefficients $\Delta_{mn}$ are called the flexibility or compliance coefficients. The relation (5) is known as Maxwell's theorem and it states that the compliance coefficients are symmetric. This is a direct consequence of Betti's theorem.
Suppose that there is a linear dependence of (generalized) displacement and (generalized) forces. Then it follows that

\[ q_n = \sum_{m=1}^{N} \Delta_{nm}Q_m = \Delta_{nm}Q_m. \]  

(6)

We assume that the linear system of equations (6) can be solved for \( Q_m \) so that

\[ Q_n = \sum_{m=1}^{N} \alpha_{nm}q_m. \]  

(7)

The quantities \( \alpha_{nm} \) are called the stiffness influence coefficients. Again, Betti's theorem (or the inverse of a symmetric matrix is symmetric) implies that \( \alpha_{nm} = \alpha_{mn} \). Note that (6) and (7) lead to

\[ \frac{\partial q_n}{\partial Q_m} = \Delta_{nm}; \quad \frac{\partial Q_n}{\partial q_m} = \alpha_{nm}. \] 

(8)

Equations (6) and (7) can be written as

\[
\begin{bmatrix}
q_1 \\
\vdots \\
q_N
\end{bmatrix} = \begin{bmatrix}
\Delta_{11} & \cdots & \Delta_{1N} \\
\vdots & \ddots & \vdots \\
\Delta_{N1} & \cdots & \Delta_{NN}
\end{bmatrix} \begin{bmatrix}
Q_1 \\
\vdots \\
Q_N
\end{bmatrix};
\]

\[
\begin{bmatrix}
Q_1 \\
\vdots \\
Q_N
\end{bmatrix} = \begin{bmatrix}
\alpha_{11} & \cdots & \alpha_{1N} \\
\vdots & \ddots & \vdots \\
\alpha_{N1} & \cdots & \alpha_{NN}
\end{bmatrix} \begin{bmatrix}
q_1 \\
\vdots \\
q_N
\end{bmatrix},
\]

(9)

with \( [\Delta_{ij}] = [\alpha_{ij}]^{-1} \).

We return now to the problem of determining \( W \) in (4). We assume that the forces \( Q_i \) are conservative. Then their work can depend only on the final state of the system and not on the way the final state is reached. This means that in integrating (4) the path of integration is unimportant. We take that all displacements are proportional to the common parameter \( \lambda \),

\[ q_n = B_n \lambda, \]  

(10)

where \( B_n \) are factors of proportionality and \( \lambda \in [0, 1] \) is a real parameter. With (10) equation (7) becomes

\[ Q_n = \sum_{m=1}^{N} \alpha_{nm}B_m \lambda. \] 

(11)

By using (10) and (11) in (4) we obtain

\[ dW = \sum_{i=1}^{N} \sum_{m=1}^{N} \alpha_{im}B_m \lambda B_i d\lambda, \] 

(12)
or
\[ W = \frac{1}{2} \sum_{i=1}^{N} \sum_{m=1}^{N} \alpha_{im} B_i B_m. \] (13)

Finally if we use (7) and (10) in (13) we obtain
\[ W = \frac{1}{2} \sum_{i=1}^{N} Q_i q_i. \] (14)

In structural mechanics (14) is called Calpeyron’s law: \textit{Work done by a system of forces acting on a linearly elastic structure is equal to one-half of the product of the final magnitudes of the generalized forces and corresponding displacements.}

We use (14) in connection with Castigiliano’s theorem.

7.5 Principle of virtual work

The Principle of virtual work belongs to the class of differential variational principles and is the basis of all variational principles in elasticity theory. It was introduced in mechanics by Bernoulli and now is taken as an \textit{axiom} that holds for all mechanical systems. Its clear statement has been given by J. R. D’Alembert (see Lindsay (1975)), and reads: \textit{in the equilibrium state of a mechanical system the work of all outer forces on virtual displacements (infinitesimal displacements that satisfy kinematical constraints) is equal to zero.}

The principle of virtual work is important for the following reasons (Marsden and Hughes 1983).

1. it is useful numerically.
2. it is believed that it remains valid under the conditions for which differential equations do not necessarily make sense.
3. Equations expressing a mathematical statement of the principle of virtual work coincide with the weak form of differential equations for which there are many relevant mathematical theorems.

The principle of virtual work is used in two different forms: \textit{the principle of virtual displacement and the principle of virtual forces}. In what follows we analyze both principles.

Next we define the concept of infinitesimal change of configuration for elastic bodies. Consider an elastic body that is in equilibrium under the action of body forces \( f \) and surface forces \( P_n \) (see Fig. 6).

We analyze the change of the configuration of the body under the assumption that this change is \textit{independent of} \( f \) \textit{and} \( P_n \). Let \( \delta u \) be the change
of the displacement vector of an arbitrary point of the body that we call the virtual displacement. This displacement field is assumed to be small independent of forces and, in principle, it may never be an actual displacement. It is a continuously differentiable vector field, independent of time and sufficiently small so that it does not influence the equilibrium of forces $f$ and $p_n$.

The displacement field $\delta u$ has an additional important property: it is a kinematically admissible field. We explain this property of $\delta u$. In the case of (see Section 4.2) third (mixed) fundamental boundary value problems in which the stress vector is prescribed on the part of the boundary $S_p$ and the displacement vector on the part of the boundary $S_u$ ($S_p \cup S_u = S$) kinematically admissible means that $\delta u(x_j) = 0$ for $x_j \in S_u$. Note that we may think of $\delta u$ as a vector field obtained from $u$ by application of the Lagrange operator of simultaneous variation $\delta (\cdot )$. When the displacement field is changed for $\delta u$ the strain tensor is changed for $\delta E_{ij}$, where

$$\delta E_{ij} = \frac{1}{2} \delta \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) = \frac{1}{2} \left( \frac{\partial \delta u_i}{\partial x_j} + \frac{\partial \delta u_j}{\partial x_i} \right). \quad (1)$$

In writing (1) we used the commutativity (see Vujanovic and Jones (1989, p.308)) of the variation operator $\delta (\cdot )$ and partial differentiation operator $\partial / \partial x_i (\cdot )$

$$\delta \left( \frac{\partial u_i}{\partial x_j} \right) = \frac{\partial \delta u_i}{\partial x_j}. \quad (2)$$

The virtual work of the forces $f$ and $p_n$ is the work of $f$ and $p_n$ on the virtual displacement $\delta u$.

### 7.6 Principle of virtual displacements

Consider a part of an elastic body shown in Fig. 7. In the deformed equilibrium configuration the resultant force acting on a volume element must
be equal to zero. Thus, we have

\[
dR = \left[ \frac{\partial}{\partial x_i} (p_i) + f \right] dV = 0,
\]

(1)

where \( p_i \) is the stress vector on a plane whose normal is the \( x_i \) axis and \( dV \) is the volume of the element. Since

\[
p_i = \sigma_{i1} e_1 + \sigma_{i2} e_2 + \sigma_{i3} e_3,
\]

(2)

the condition \( dR = 0 \) leads to the equilibrium equations (1.3-10). Suppose that the point of application of the force \( dR \) is subjected to virtual displacement \( \delta u \). Then the virtual work reads

\[
dR \cdot \delta u = \left( \frac{\partial \sigma_{ij}}{\partial x_j} + f_i \right) \delta u_i = 0.
\]

(3)

By using the expression

\[
\frac{\partial \sigma_{ij}}{\partial x_j} \delta u_i = \frac{\partial}{\partial x_j} (\sigma_{ij} \delta u_i) - \sigma_{ij} \frac{\partial \delta u_i}{\partial x_j},
\]

(4)

and the following consequence of the symmetry of the stress tensor and (7.5-2)

\[
\sigma_{ij} \frac{\partial \delta u_i}{\partial x_j} = \sigma_{ij} \left( \frac{\partial \delta u_i}{\partial x_j} + \frac{\partial \delta u_j}{\partial x_i} \right) = \sigma_{ij} \delta E_{ij},
\]

(5)

in equation (3), we obtain, after integration over the volume \( V \) of the body,

\[
\int \int \int_V \frac{\partial}{\partial x_j} (\sigma_{ij} \delta u_i) dV + \int \int \int_V \delta u_i dV = \int \int \int_V \sigma_{ij} \delta E_{ij} dV.
\]

(6)

The first term on the left-hand side could be transformed by using Gauss’ theorem, so that

\[
\int \int_{S_p} p_i \delta u_i dS + \int \int_V f_i \delta u_i dV = \int \int_V \sigma_{ij} \delta E_{ij} dV.
\]

(7)
The expression on the left-hand side is the virtual work of outer forces $p_n$ and $f$, whereas the expression on the right-hand side is the virtual work of the inner forces, that is, stresses. By setting

$$\delta W^o = \int_{S_p} p_i \delta u_i dS + \int_V \int f_i \delta u_i dV; \quad \delta W^i = \int_V \int \sigma_{ij} \delta E_{ij} dV,$$

we can write (7) as

$$\delta W^o = \delta W^i. \quad (8)$$

When the principle of virtual work is taken as an axiom then (9) is postulated (instead of (1)). The relation (9) is called the principle of virtual displacements and is stated as: for a body in equilibrium the total virtual work of outer forces (surface and body) is equal to the total virtual work of inner forces for any kinematically admissible virtual displacement field.

As stated in the form (9) the principle of virtual displacement is valid for all materials (elastic, viscoelastic, plastic, etc.). When the elastic materials are analyzed the virtual work of inner forces becomes (see (7.2-17)) the virtual change of the potential energy of the body; that is,

$$\delta W_i = \delta U. \quad (10)$$

The principle of virtual displacement could be used in dynamic problems if we replace $f_i$ by $f_i - \rho_0 (\partial^2 u_i / \partial t^2)$. Then, equation (9) remains the same and $\delta W^o$ becomes

$$\delta W^o = \int_{S_p} p_i \delta u_i dS + \int_V \int \left( f_i - \rho_0 \frac{\partial^2 u_i}{\partial t^2} \right) \delta u_i dV. \quad (11)$$

For elastic materials (11) becomes

$$\int_{S_p} p_i \delta u_i dS + \int_V \int \left( f_i - \rho_0 \frac{\partial^2 u_i}{\partial t^2} \right) \delta u_i dV = \int_V \int \sigma_{ij} \delta E_{ij} dV. \quad (12)$$

### 7.7 Principle of virtual forces

The principle of virtual forces is a complementary principle of the principle of virtual displacements. It is also called the principle of virtual complementary work. In it we determine the virtual pseudowork that is done by virtual inner and outer forces on the real displacements that the body has in given equilibrium configurations. The virtual inner and outer forces are arbitrary sufficiently small fields that do not depend on the actual surface, body, and inner (stresses) forces.
We start from the equations

$$\frac{\partial \sigma_{ij}}{\partial x_j} + f_i = 0; \quad \sigma_{ij} n_j = p_i, \quad \text{on } S_p,$$  

(1)

that are satisfied in an equilibrium configuration. We introduce the virtual stresses and virtual surface and body forces \(\delta \sigma_{ij}, \delta p_i,\) and \(\delta f_i.\) The fields \(\delta \sigma_{ij}, \delta p_i,\) and \(\delta f_i\) are assumed to be continuously differentiable and statically admissible, that is, the field satisfies \(\delta \sigma_{ij} = \delta \sigma_{ji}\) and

$$\frac{\partial \delta \sigma_{ij}}{\partial x_j} + \delta f_i = 0;$$

$$\delta \sigma_{ij} n_j = 0 \quad \text{on } S_p \quad \delta \sigma_{ij} n_j = \delta p_i \quad \text{on } S_u. \quad (2)$$

For the case when the body forces are prescribed, we take \(\delta f_i = 0.\) The functions \(\delta p_i\) are arbitrary. Note that the stress tensor components \(\sigma_{ij}\) satisfy Beltrami-Michell compatibility conditions and that the virtual stresses \(\delta \sigma_{ij}\) do not satisfy any such condition. By taking the scalar product of (2) by \(u\) and by performing integration of the result over \(V,\) we obtain

$$\int \int_V \int \left( \frac{\partial \delta \sigma_{ij}}{\partial x_j} + \delta f_i \right) u_i dV = 0. \quad (3)$$

However

$$u_i \frac{\partial \delta \sigma_{ij}}{\partial x_j} = \frac{\partial (u_i \delta \sigma_{ij})}{\partial x_j} - \frac{\partial u_i}{\partial x_j} \delta \sigma_{ij}, \quad (4)$$

so that by substituting (4) in (3) we obtain

$$\int \int_S u_i \delta p_i dS - \int \int_V \int \frac{\partial u_i}{\partial x_j} \delta \sigma_{ij} dV + \int \int_V \int u_i \delta f_i dV = 0. \quad (5)$$

In writing (5) we used the Gauss theorem. In (5) the part

$$\delta W_c^o = \int \int_S u_i \delta p_i dS + \int \int_V \int u_i \delta f_i dV, \quad (6)$$

represents the virtual (pseudo) work of the outer forces. Note also that because of (2) we have

$$\int \int_S u_i \delta p_i dS = \int \int_{S_u} u_i \delta p_i dS. \quad (7)$$

Since the virtual stress components \(\delta \sigma_{ij}\) are symmetric we can transform the middle term in (5) as

$$\delta W_c^i = \int \int_V \int \frac{\partial u_i}{\partial x_j} \delta \sigma_{ij} dV = \int \int_V \int E_{ij} \delta \sigma_{ij} dV. \quad (8)$$
The expression (8) represents the (pseudo) virtual work of the inner forces. With (6) and (8) equation (5) reads

\[ \delta W^i = \delta W^o. \] (9)

The relation (9) is the mathematical expression of the principle of virtual forces. It is stated as: the displacement field \( u_i \) and the strain tensor \( E_{ij} \) that satisfy boundary and compatibility conditions correspond to the equilibrium configuration of the body if the virtual complementary work of inner and outer forces is equal.

Note that for the case of an elastic body we have the virtual complementary work given by (7.2-19) so that

\[ \delta U_c = \delta W^o. \] (10)

We stress now an important characteristic of the results obtained here. Namely, by comparing (7.2-3) and (7.6-10) we conclude that those relations are similar. However, there exists a fundamental difference between (7.2-3) and (7.6-11). The condition (7.2-3) connects the change of the potential energy with the work of outer forces between these two equilibrium configurations of the body. Therefore the vector \( du \) connecting these two equilibrium configurations is known. The relation (7.6-10) is not a relation valid for two equilibrium configurations. Instead it defines an equilibrium configuration. Thus the vector \( \delta u \) is not known. We only know that it satisfies conditions stated in Section 7.5. Then if (7.6-10) holds for arbitrary \( \delta u \) the configuration is equilibrium one. Similar analysis holds for (7.2-5) and (7.7-10).

7.8 Minimum of potential and complementary energy theorems

On the basis of results obtained so far we can formulate two important extremal principles of elasticity theory. We start with

i) The theorem of minimum of the potential energy

From equation (7.6-11) we obtain

\[ \delta_u(U - W^i) = 0, \] (1)

where the operator \( \delta_u(\cdot) \) indicates variation of the displacement field \( u \) while \( p \) and \( f \) (since they are prescribed) are not subject to variation. By using (7.6-7) in (1) we have

\[ \delta_u \left( U - \int_{s_p} p_i u_i dS - \int_V \int f_i u_i dV \right) = 0. \] (2)
If we define the *total potential energy* (of inner and outer forces) as

\[ \Pi = U - \int_S \int_{S_p} p_i u_i dS - \int_V \int f_i u_i dV, \]  

(3)

then (1) becomes

\[ \delta_u \Pi = 0. \]  

(4)

Equation (4) states that \( \Pi \) has a stationary value in the class of admissible (virtual) displacements \( \delta u_i \). The condition (4) characterizes the equilibrium states of an elastic body. In using (4) we have to substitute the constitutive equation \( \sigma_{ij} = \sigma_{ij}(E_{km}) \). Thus by using (7.2-27) we have

\[
\begin{align*}
\Pi &= \int_V \int \left[ \frac{1}{2} \lambda (E_{nn})^2 + \mu E_{ij} E_{ij} \right] dV - \int_S \int_{S_p} p_i u_i dS - \int_V \int f_i u_i dV \\
&= \int_V \int \left[ \frac{1}{2} \lambda (E_{11} + E_{22} + E_{33})^2 + \mu (E_{11}^2 + E_{22}^2 + E_{33}^2) \right] dV - \int_S \int_{S_p} p_i u_i dS - \int_V \int f_i u_i dV.
\end{align*}
\]

(5)

The functional (5) has a stationary value in the equilibrium configuration. We show next that the functional \( \Pi \) assumes a minimum in the equilibrium state. To do this, we proceed as follows. Let \( \delta u \) be an admissible (virtual) displacement field. Let \( E_{ij} \) be the strain tensor in the equilibrium state and let \( E_{ij} + \delta E_{ij} \) be the strain tensor in the varied configuration. The total potential energy \( \Pi \) (given by (5)) in the varied state becomes

\[
\begin{align*}
\bar{\Pi} &= \int_V \int \left[ \frac{1}{2} \lambda (E_{nn} + \delta E_{nn})(E_{nn} + \delta E_{nn}) + \mu (E_{ij} + \delta E_{ij}) \right. \\
&\times (E_{ij} + \delta E_{ij}) dV - \int_S \int_{S_p} p_i (u_i + \delta u_i) dS - \int_V \int f_i (u_i + \delta u_i) dV.
\end{align*}
\]

(6)

By rearranging terms in (6) we get

\[
\begin{align*}
\bar{\Pi} &= \int_V \int \left[ \frac{1}{2} \lambda (E_{nn})^2 + \mu E_{ij} E_{ij} - f_i u_i \right] dV - \int_S \int_{S_p} p_i u_i dS \\
&+ \int_V \int \left[ (\lambda \delta_{mn} E_{rr} + 2 \mu E_{mn}) \delta E_{mn} - f_i \delta u_i \right] dV - \int_S \int_{S_p} p_i \delta u_i dS \\
&+ \int_V \int \left[ 2 \mu \delta E_{mn} \delta E_{mn} + \lambda (\delta E_{rr})(\delta E_{rr}) \right] dV.
\end{align*}
\]

(7)
Using (4), (5) in (7) we obtain
\[ \Pi = \Pi + \int \int_V \int [2\mu \delta E_{mn} \delta E_{mn} + \lambda (\delta E_{rr})^2]dV. \]  
(8)

Since the expression in brackets is positive definite (\( \lambda > 0, \mu > 0 \)) it follows that
\[ \Pi > \Pi. \]  
(9)

With (9) we can state the following theorem of the minimum of the potential energy: of all displacement fields satisfying the boundary conditions those that satisfy equilibrium conditions make the potential energy \( \Pi \) minimal.

ii) The theorem of minimum of the complementary energy

From equation (7.7-10) we have
\[ \delta_F (U_c - W^0) = 0. \]  
(10)

In (10) the operator \( \delta_F (\cdot) \) denotes the variation of the force field \( \sigma_{ij}, p_i, \) and \( f_i \) while \( u_i \) is kept fixed. If we define the total complementary energy as
\[ \Pi_c = U_c - \int S_u u_i p_i dS - \int V f_i dV, \]  
(11)

then (10) becomes
\[ \delta_F \Pi_c = 0. \]  
(12)

For a linearly elastic body we use (7.2-19), (7.2-25) in (11) so that
\[ \Pi_c = \int V \int \frac{1}{2E} [(1 + \nu)\sigma_{ij} \sigma_{ij} - \nu (\sigma_{rr})^2]dV \]
\[ - \int S_u u_i p_i dS - \int V f_i u_i dV \]
\[ = \int V \int \frac{1}{2E} [(\sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2) - 2\nu (\sigma_{11} \sigma_{22} + \sigma_{11} \sigma_{33} + \sigma_{22} \sigma_{33})] \]
\[ + 2(1 + \nu)(\sigma_{12}^2 + \sigma_{13}^2 + \sigma_{23}^2)] \]
\[ - \int S_u u_i p_i dS - \int V f_i u_i dV. \]  
(13)

The condition (12) implies that \( \Pi_c \) given by (13) is stationary in the equilibrium configuration. Let \( \delta \sigma_{ij}, \delta p_i, \) and \( \delta f_i \) be a variation of the force
field. Then we determine $\bar{\Pi}_c = \Pi(\sigma_{ij} + \delta\sigma_{ij}, p_i + \delta p_i, f_i + \delta f_i)$ in the following form

$$\bar{\Pi}_c = \int \int \int_V \frac{1}{2E} [(1 + \nu)\sigma_{ij}\sigma_{ij} - \nu(\sigma_{rr})^2]dV$$

$$- \int \int_{S_u} u_ip_i dS - \int \int_{V} f_i u_i dV - \int \int_{S_u} u_i \delta p_i dS - \int \int_V \delta f_i u_i dV$$

$$+ \int \int_V \int \frac{1}{E} \left[ \frac{1 + \nu}{E} \sigma_{ij} - \frac{\nu}{E} (\sigma_{mm}) \delta_{ij} \right] \delta\sigma_{ij} dV$$

$$+ \int \int_V \int \frac{1}{2E} [(1 + \nu)\delta\sigma_{ij}\delta\sigma_{ij} - \nu(\delta\sigma_{rr})^2]dV. \quad (14)$$

Note that the third and fourth term in (14) are just $\delta_F\Pi_c = 0$ so that

$$\bar{\Pi}_c - \Pi_c = \int \int \int_V \frac{1}{2E} [(1 + \nu)\delta\sigma_{ij}\delta\sigma_{ij} - \nu(\delta\sigma_{rr})^2]dV \geq 0. \quad (15)$$

The equality sign in (15) holds for $\delta\sigma_{ij} = 0$ only. Thus we can state the following theorem of the minimum of complementary energy: the complementary energy $\Pi_c$ has a minimum when the stress tensor $\sigma_{ij}$ satisfies equilibrium conditions and variations of the force field $\delta\sigma_{ij}, \delta p_i, \text{and} \delta f_i$ satisfy (7.7-2).

Note that the conditions (7.2-2) are used through $\delta_F\Pi_c = 0$ in writing (15).

We now give two examples of the variational principles formulated in this Section. Consider the plane state of stress (see (6.1-15)). Then (13) becomes

$$\Pi_c = \frac{1}{2E} \int \int_V \int [\sigma_{11}^2 + \sigma_{22}^2 - 2\nu\sigma_{11}\sigma_{22} + 2(1 + \nu)\sigma_{12}^2]dV$$

$$- \int \int_{S_u} u_ip_i dS - \int \int_{V} f_i u_i dV. \quad (16)$$

As a next example, consider a plate of unit thickness. Suppose that the body force is equal to zero and that the stress vector is prescribed on the boundary of the plate ($S = S_p$). If we introduce a stress function $\Phi$ by (6.2-4) with $V = 0$, we obtain

$$\Pi_c = \frac{1}{2E} \int_A \left[ \left( \frac{\partial^2\Phi}{\partial x_1^2} \right)^2 + \left( \frac{\partial^2\Phi}{\partial x_2^2} \right)^2 - 2\nu \frac{\partial^2\Phi}{\partial x_1^2} \frac{\partial^2\Phi}{\partial x_2^2} \right. \right.$$

$$+ 2(1 + \nu) \left. \left( \frac{\partial^2\Phi}{\partial x_1 \partial x_2} \right)^2 \right] dA, \quad (17)$$
where $A$ is the area of the middle plane $B$. The boundary conditions are (see (6.2-9))

$$
\frac{\partial \Phi}{\partial x_1} = - \int_0^S \bar{p}_2(u) du = \psi_1(S) + C_1; \\
\frac{\partial \Phi}{\partial x_2} = \int_0^S \bar{p}_1(u) du = \psi_2(S) + C_2,
$$

on $C$, where $C$ is the boundary of $B$ (see Fig. 1 in Section 6).

### 7.9 Castigliano's theorems

We now present two important theorems of structural mechanics. They follow from the results of Section 7.8.

Consider the expression (7.8-5) for the case when the body forces are equal to zero and there are only concentrated forces acting on the boundary of the body. The total potential energy then becomes

$$
\Pi = \int \int_V \int \left[ \frac{1}{2} \lambda (E_{nn})^2 + \mu E_{ij} E_{ij} \right] dV - \sum_{i=1}^N Q_i q_i, 
$$

where $Q_i$ are generalized forces and $q_i$ displacements of the point of the application of the force $Q_i$. We assume that $E_{ij}$ could be written in terms of $q_i$ so that (1) becomes

$$
\Pi = U(q_i) - \sum_{i=1}^N Q_i q_i. 
$$

The condition of the minimum of the potential energy $\delta_u \Pi = 0$ (see (7.8-4)) becomes

$$
\delta_u \Pi = \sum_{i=1}^N \frac{\partial \Pi}{\partial q_i} \delta q_i = 0. 
$$

By using (2) in (3) and the fact that $\delta q_i$ are arbitrary, we obtain

$$
Q_i = \frac{\partial U}{\partial q_i}. 
$$

The result (4) is known as Castigliano's first theorem: in a linearly elastic system (the generalized force depends linearly on the generalized displacement) the partial derivative of a potential energy of inner forces (strain energy) with respect to the generalized displacement is equal to the generalized force.
Suppose again that the body forces are equal to zero and that the surface forces are concentrated and act at \( i = 1, 2, \ldots, N \) points. In this case (7.8-13) becomes

\[
P_c = \int \int_V \int \frac{1}{2E} [(1 + \nu)\sigma_{ij}\sigma_{ij} - \nu(\sigma_{rr})^2]dV - \sum_{i=1}^{N} q_i Q_i.
\]  

If we express the stresses in terms of \( Q_i \) and use this in (5) we obtain

\[
P_c = U_c(Q_i) - \sum_{i=1}^{N} q_i Q_i.
\]  

The condition that \( P_c \) is minimal (7.8-12) reads

\[
\delta F P_c = \sum_{i=1}^{N} \left( \frac{\partial U_c}{\partial Q_i} - q_i \right) \delta Q_i = 0.
\]  

Since \( \delta Q_i \) are arbitrary it follows that

\[
q_i = \frac{\partial U_c}{\partial Q_i}.
\]  

The expression (8) is the second Castigliano theorem: in a linearly elastic system the partial derivative of the strain energy with respect to an externally applied generalized force is equal to the displacement corresponding to that force.

For the case when \( q_i = 0 \) the expression is known as the Menabrea law.

### 7.10 Hu-Washizy and Reissner variational principles

In connection with finite element procedures other variational principles are used. We present two such principles.

1. **Hu-Washizy variational principle**

A generalized variational principle in which three fields \( u_i, E_{ij}, \) and \( \sigma_{ij} \) are subject to variation is formulated in Washizu (1968). We consider a mixed boundary value problem with the stress vector prescribed on the part \( S_p \) of the boundary and displacement prescribed on the part \( S_u \) of the boundary (see (4.2-5))

\[
(p_n)_i = \sigma_{ij}n_j = \hat{p}_i, \quad \text{on } S_p, \quad u_i(x_i) = \hat{u}_i \quad \text{on } S_u.
\]
Consider the following functional

\[
I_W(u_i, E_{ij}, \sigma_{ij}) = \int \int_V \int [u(E_{ij}) - f_i u_i] dV \\
- \int \int_V \int \sigma_{ij} \left[ E_{ij} - \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right] dV \\
- \int_{S_p} \hat{p}_i u_i dS - \int_{S_u} \sigma_{ij} n_j (u_i - \hat{u}_i) dS.
\]  

(2)

In (2) we consider \( u_i, E_{ij}, \) and \( \sigma_{ij} \) as independent fields. Therefore the variations \( \delta u_i, \delta E_{ij}, \) and \( \delta \sigma_{ij} \) are independent. Also \( u(E_{ij}) \) is given by (7.2-27); that is,

\[
u = \frac{1}{2} \lambda (E_{nn})^2 + \mu E_{ij} E_{ij}. \quad (3)
\]

The condition that the first variation of (2) is equal to zero becomes

\[
\delta I_W = \int \int_V \int \left[ \frac{\partial u}{\partial E_{ij}} \delta E_{ij} - f_i \delta u_i \right] dV \\
- \int \int_V \int \delta \sigma_{ij} \left[ E_{ij} - \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right] dV \\
- \int \int_V \int \sigma_{ij} \left[ \delta E_{ij} - \frac{1}{2} \left( \frac{\partial \delta u_i}{\partial x_j} + \frac{\partial \delta u_j}{\partial x_i} \right) \right] dV \\
- \int_{S_p} \hat{p}_i \delta u_i dS - \int_{S_u} \delta \sigma_{ij} n_j (u_i - \hat{u}_i) dS = 0. \quad (4)
\]

By rearranging terms in (4) we obtain

\[
\int \int_V \int \left[ \frac{\partial u}{\partial E_{ij}} - \sigma_{ij} \right] \delta E_{ij} dV - \int \int_V \int \left[ \frac{\partial \sigma_{ij}}{\partial x_j} - f_i \right] dV \\
- \int \int_V \int \delta \sigma_{ij} \left[ E_{ij} - \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right] dV \\
- \int_{S_p} (\hat{p}_i - p_i) \delta u_i dS - \int_{S_u} \delta p_i (u_i - \hat{u}_i) dS = 0. \quad (5)
\]

From (5) it follows that

\[
\frac{\partial u}{\partial E_{ij}} - \sigma_{ij} = 0; \quad \frac{\partial \sigma_{ij}}{\partial x_j} + f_i = 0; \quad E_{ij} - \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) = 0, \quad (6)
\]

in \( V \) and

\[
\sigma_{ij} n_j - \hat{p}_i = 0 \quad \text{on} \quad S_p, \quad u_i - \hat{u}_i = 0, \quad \text{on} \quad S_u. \quad (7)
\]
From (6)_1 we conclude that \( \partial u/\partial E_{ij} = \sigma_{ij} \) and from (3) it follows that (6)_1 represents Hooke's law, (6)_2 equilibrium equations, and (6)_3 the strain displacement relation, whereas (7) represents boundary conditions. On the basis of this we can state the following theorem: Hu-Washizu functional (2) is stationary on the solution of mixed boundary value problems of linear elasticity theory.

Note that variational principle \( \delta I_W = 0 \) is just a stationarity principle and that it has no extremal properties.

2. Reissner variational principle

Reissner (1950) formulated a variational principle where two fields are independent. We consider again the mixed boundary value problem (1) and in connection with it the functional

\[
I_R(u_i, \sigma_{ij}) = \int \int_V \left[ \frac{1}{2} \sigma_{ij} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - u_c(\sigma_{ij}) - f_i u_i \right] dV
- \int \int_{S_p} \hat{p}_i u_i dS - \int \int_{S_u} \sigma_{ij} n_j (u_i - \dot{u}_i) dS.
\]

In (8) \( u_c \) the complementary energy density is given by (7.2-24); that is,

\[
u_c = \frac{1}{2E} \left[ (1 + \nu) \sigma_{mn} \sigma_{mn} - \nu \Theta^2 \right].
\]

By calculating the first variation of (9) taking \( \delta u_i \) and \( \delta \sigma_{ij} \) as independent, we obtain

\[
\delta I_R(u_i, \sigma_{ij}) = \int \int_V \left[ \frac{1}{2} \delta \sigma_{ij} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \frac{1}{2} \sigma_{ij} \left( \frac{\partial \delta u_i}{\partial x_j} + \frac{\partial \delta u_j}{\partial x_i} \right) \right.
- \frac{\partial u_c(\sigma_{ij})}{\partial \sigma_{ij}} \delta \sigma_{ij} - f_i \delta u_i] dV
- \int \int_{S_p} \hat{p}_i \delta u_i dS - \int \int_{S_u} \delta \sigma_{ij} n_j (u_i - \dot{u}_i) dS = 0.
\]

Note that

\[
\frac{1}{2} \delta \sigma_{ij} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) = \delta \sigma_{ij} \frac{\partial u_i}{\partial x_j}; \quad \frac{1}{2} \sigma_{ij} \left( \frac{\partial \delta u_i}{\partial x_j} + \frac{\partial \delta u_j}{\partial x_i} \right) = \sigma_{ij} \frac{\partial \delta u_i}{\partial x_j};
\]

\[
\frac{\partial u_c}{\partial \sigma_{ij}} = E_{ij},
\]

as a result of symmetry of \( \delta \sigma_{ij} \) and \( \delta E_{ij} \) and the definition of \( u_c \). By using (11) and the Gauss theorem in (10) we have

\[
\frac{\partial u_c}{\partial \sigma_{ij}} - E_{ij} = 0; \quad \frac{\partial \sigma_{ij}}{\partial x_j} + f_i = 0.
\]
in \( V \) and
\[
\sigma_{ij} n_j - \hat{p}_i = 0; \quad \text{on } S_p, \quad u_i - \hat{u}_i = 0 \quad \text{on } S_u. \tag{13}
\]

From (12) we can state the following theorem: *Reissner functional (8) is stationary; that is, \( \delta I_R = 0 \) on the solution of mixed boundary value problems of linear elasticity theory.*

Both the Hu-Washizy and Reissner variational principles have their counterpart in nonlinear elasticity (see Lurie (1980)).

3. Operator form of variational principles

We make one more remark concerning the variation principles formulated in this section. When used in connection with the finite element method these principles are written in a slightly different form that follows from the operator representation of basic equations of elasticity theory. To obtain this form we first write the operator form of the equations of elasticity theory.

Let \( \sigma, \mathbf{E}, \mathbf{u}, \) and \( \mathbf{f} \) be the vectors of stress, strain, displacement, and body forces defined as
\[
\sigma = [\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{23}, \sigma_{13}]^T; \quad \mathbf{E} = [E_{11}, E_{22}, E_{33}, E_{12}, E_{23}, E_{13}]^T;
\]
\[
\mathbf{u} = [u_1, u_2, u_3]^T; \quad \mathbf{f} = [f_1, f_2, f_3]^T,
\tag{14}
\]
where \([\cdot]^T\) denotes the transpose of \([\cdot]\). Let \( \mathbf{C} \) be the elasticity matrix defined earlier in Section 3.3 (see (3.3-42)). For isotropic materials it has the form
\[
\mathbf{C} = \begin{bmatrix}
\lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\
\lambda & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\
\lambda & \lambda & \lambda + 2\mu & 0 & 0 & 0 \\
0 & 0 & 0 & \mu & 0 & 0 \\
0 & 0 & 0 & 0 & \mu & 0 \\
0 & 0 & 0 & 0 & 0 & \mu \\
\end{bmatrix}. \tag{15}
\]

Next we introduce the matrix operators defined as
\[
\mathbf{A} = \begin{bmatrix}
D_1 & 0 & 0 \\
0 & D_2 & 0 \\
0 & 0 & D_3 \\
D_2 & D_1 & 0 \\
0 & D_3 & D_2 \\
D_3 & 0 & D_1 \\
\end{bmatrix}; \quad \mathbf{A}_S = \begin{bmatrix}
n_1 & 0 & 0 \\
0 & n_2 & 0 \\
0 & 0 & n_3 \\
n_2 & n_1 & 0 \\
0 & n_3 & n_2 \\
n_3 & 0 & n_1 \\
\end{bmatrix}. \tag{16}
\]

where \( D_i = \partial (\cdot) / \partial x_i \). Further, let \( D_{ij} \) denote the differential operator
\[
D_{ij}(\cdot) = \frac{\partial^2}{\partial x_i \partial x_j} (\cdot). \tag{17}
\]
Then the compatibility conditions ((2.7-13) with (2.9-4)) could be written in compact form if we introduce the matrix differential operator $B$ by

$$
B = \begin{bmatrix}
D_{22} & D_{11} & 0 & -D_{12} & 0 & 0 \\
0 & D_{33} & D_{22} & 0 & -D_{23} & 0 \\
D_{33} & 0 & D_{11} & 0 & 0 & -D_{13} \\
D_{23} & 0 & 0 & -\frac{1}{2}D_{13} & \frac{1}{2}D_{11} & -\frac{1}{2}D_{12} \\
0 & D_{13} & 0 & -\frac{1}{2}D_{23} & -\frac{1}{2}D_{12} & \frac{1}{2}D_{11} \\
0 & 0 & \frac{1}{2}D_{12} & \frac{1}{2}D_{33} & -\frac{1}{2}D_{12} & -\frac{1}{2}D_{23}
\end{bmatrix}.
$$

(18)

With (13) through (18) we can state the basic equations of elasticity theory as:

1. **Equilibrium equations**

   $$
   A^T \sigma + g = 0.
   $$

   (19)

2. **Strain displacement relations**

   $$
   E = Au.
   $$

   (20)

3. **Hooke’s law and inverse Hooke’s law**

   $$
   \sigma = CE \quad \text{or} \quad E = C^{-1}\sigma.
   $$

4. **Boundary conditions**

   $$
   A_S \sigma = \hat{p} \quad \text{on} \quad S_p, \quad u = \hat{u} \quad \text{on} \quad S_u.
   $$

   (21)

5. **Compatibility conditions**

   $$
   BE = 0.
   $$

   (22)

Note that the strain tensor given by (20) identically satisfies the compatibility conditions (22) since

$$
BA = 0.
$$

(23)

In the case when body forces are equal to zero the Beltrami-Michell compatibility conditions (3.7-21) have the form

$$
\nabla^2 \sigma_{ij} + \frac{1}{1 + \nu} \frac{\partial^2 \Theta}{\partial x_i \partial x_j} = 0.
$$

(24)

Introducing the operator $L$ by

$$
L = \begin{bmatrix}
D_{11} & 0 & 0 & 0 & 0 & 0 \\
0 & D_{22} & 0 & 0 & 0 & 0 \\
0 & 0 & D_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & D_{12} & 0 & 0 \\
0 & 0 & 0 & 0 & D_{23} & 0 \\
0 & 0 & 0 & 0 & 0 & D_{13}
\end{bmatrix},
$$

(25)
equation (23) could be written as

\[ \nabla^2 \sigma + \frac{1}{1 + \nu} L \Theta = 0 . \quad (26) \]

With the notation (13) through (26) the variational principles defined in this section become

1. The theorem of the minimum of the potential energy is stated as

\[ \delta_u \Pi = 0 , \quad (27) \]

where

\[ \Pi(u) = \frac{1}{2} \int \int_V (Au)^T C(Au) dV - \int \int_V \int u^T f dV - \int \int_{S_p} u^T \hat{p} dS . \quad (28) \]

2. The theorem of the minimum of complementary energy is

\[ \delta_F \Pi_c = 0 , \quad (29) \]

where

\[ \Pi_c(\sigma, p, f) = -\frac{1}{2} \int \int_V \int \sigma^T |C|^{-1} \sigma dV - \int \int_V \int f^T u dV + \int \int_{S_p} u^T (A_S)^T \sigma dS . \quad (30) \]

3. The Hu-Washizy variational principle becomes

\[ \delta I_W = 0 , \quad (31) \]

where

\[ I_W(u, E, \sigma) = \int \int_V \left[ \frac{1}{2} E^T CE - u^T f \right] dV - \int \int_V \int \sigma^T [E - Au] dV - \int \int_{S_p} u \hat{p} dS - \int \int_{S_u} [u - \hat{u}]^T (A_S)^T \sigma dS . \quad (32) \]

4. The Reissner variational principle is

\[ \delta I_R = 0 , \quad (33) \]

where

\[ I_R(u, \sigma) = \int \int_V \int [(Au)^T \sigma - \sigma C^{-1} \sigma^T - fu^T] dV - \int \int_{S_p} \hat{p} u^T dS - \int \int_{S_u} A_S \sigma (u - \hat{u}) dS . \quad (34) \]

For details concerning variational principles presented here and possible generalizations see Oden and Reddy (1974, 1983) and Rozin (1978).
Problems

1. Show that for the case of a torsion of a prismatic rod the complementary energy (7.8-13) has the form

\[ \Pi_c = \frac{1 + \nu}{2E} \int \int_V \left( \sigma_{12}^2 + \sigma_{13}^2 \right) dV - \int_A (p_2 u_2 + p_3 u_3) dA, \]

where \( A \) is the cross-sectional area of the rod. Furthermore, by using (5.5-31) show that \( \Pi_c \) could be written as

\[ \Pi_c = \frac{\mu \bar{\theta}^2 l}{4} \int \int_A \left[ \left( \frac{\partial \Psi}{\partial x_2} \right)^2 + \left( \frac{\partial \Psi}{\partial x_3} \right)^2 + 2 \left( x_2 \frac{\partial \Psi}{\partial x_2} + x_3 \frac{\partial \Psi}{\partial x_3} \right) \right] dA, \]

where \( l \) is the length of the rod.

2. Show that \( \Pi_c \) obtained in Problem 1 could be expressed as

\[ \Pi_c = \frac{\mu \bar{\theta}^2 l}{2} \int \int_A \left[ \left( \frac{\partial \Psi}{\partial x_2} \right)^2 + \left( \frac{\partial \Psi}{\partial x_3} \right)^2 - 4 \Psi \right] dA. \]  

(a)

Use the Ritz method (see Section 4.6) to obtain an analytical approximation of the stress function for the square cross-section (see (5.5-9)). Assume the trial solution of the form

\[ \Psi = c(x_2^2 - a^2)(x_3^2 - a^2), \]  

(b)

where \( c \) is a constant. Then substitute (b) into (a) and minimize the result with respect to \( c \). From the condition \( d\Pi_c/dc = 0 \) obtain \( c = 5/(8a^2) \). The approximate value of the resultant couple given by (5.5-37) is

\[ M = 2\mu \bar{\theta} \int \int_A \Psi dA ; \]

with (b) this becomes

\[ M = \frac{20}{9} \mu \bar{\theta} a^4 = 0.1388(2a)^4 \mu \bar{\theta} . \]  

(c)

The result agrees well with the exact solution ((5.5-106) with \( a = b \). Show that the more complicated trial function

\[ \Psi = (x_2^2 - a^2)(x_3^2 - a^2)[c_0 + c_1(x_1^2 + x_2^2)], \]  

(d)

where \( c_0 \) and \( c_1 \) are constants, together with \( d\Pi_c/dc_0 = 0, d\Pi_c/dc_1 = 0 \), leads to an improved solution.
3. By using Betti’s theorem (7.3-8) and two special systems of forces

\[
\begin{align*}
    f_i^I &= 0; & \sigma_{ij}^I &= \delta_{ij}; & p_i^I &= n_i; & u_i^I = f_i^I \\
    f_i^I &= f_i; & \sigma_{ij}^I &= \sigma_{ij}; & p_i^I &= p_i; & u_i^I = u_i,
\end{align*}
\]

show that

\[
\int_s \int n_i u_i dS = \int_s \int p_i u_i^I dS + \int_V \int f_i u_i^I dV.
\]

4. In Problem 3 use (3.4-14) and show that for the system of forces I we have

\[
E_{ij}^I = \frac{1 - 2\nu}{E} \sigma_{ij}, \quad u_i^I = \frac{1 - 2\nu}{E} x_i.
\]

By using this result and Gauss’ theorem, show that the change of volume $\Delta V$ in an elastic body under the action of surface forces $p_i$ and the body forces $f_i$ is (see Sokolnikoff (1956 p. 395))

\[
\Delta V = \int_V \int eV dV = \frac{1 - 2\nu}{E} \left( \int_s \int p_i x_i dS + \int_V \int f_i x_i dV \right).
\]

5. Two elastic rods of length $L$ of cross-sectional area $A$ and made of a material whose modulus of elasticity is $E$ are connected by joints at the points B, C, and D. At the point C a concentrated force of intensity $F$ is applied (see figure).

\[
U = \frac{1}{2} \int_V \int \sigma e dV = \int_V \int \frac{F \Delta}{A L} dV
\]

\[
= \int_0^L \left[ \int_A E \left( \frac{\Delta}{L} \right)^2 dA \right] dz = \frac{EA \Delta^2}{L}.
\]
ii) The elongation $\Delta$ of the rod and the vertical deflection of the midpoint $C$, denoted in the figure by $q$, are connected by

$$\Delta = \sqrt{L^2 + q^2} - L,$$

so that

$$\Delta = \frac{q^2}{2L} + \ldots \quad \text{(b)}$$

iii) By combining $(a)$ and $(b)$ prove that

$$U = \frac{EAq^4}{4L^2},$$

so that Castigiliano’s first theorem (7.9-4) leads to

$$F = \frac{\partial U}{\partial q} = \frac{EAq^3}{L^3}. \quad \text{(c)}$$

The result $(c)$ shows that in an elastic system the force displacement relation can be nonlinear even in the case when the stress-strain law (constitutive relation) is linear.

6. Consider the functional (7.8-5) for the case when the boundary is stress free. Show that it could be written in the form

$$\Pi = \int \int_{V} \int \left[ \frac{1}{2} \lambda (E_{nn})^2 + \mu E_{ij} E_{ij} - f_i u_i \right] dV$$

$$= \int \int_{V} \int \left[ \frac{\mu}{4} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \frac{\lambda}{4} \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i} + f_i u_i \right] dV$$

$$= \int \int_{V} \int L \left( u_i, \frac{\partial u_i}{\partial x_j} \right) dV. \quad \text{(a)}$$

i) Write the Euler-Lagrange equations for $(a)$; that is,

$$\frac{\partial}{\partial x_m} \left( \frac{\partial L}{\partial \left( \frac{u_i}{\partial x_m} \right)} \right) - \frac{\partial L}{\partial u_i} = 0 \quad \text{(b)}$$

and show that $(b)$ agrees with Lamé equations (4.1-8); that is,

$$(\lambda + \mu) \frac{\partial}{\partial x_i} \left( \frac{\partial u_j}{\partial x_j} \right) + \mu \nabla^2 u_i + f_i = 0. \quad \text{(c)}$$
Chapter 8

Elementary Theory of Plates

8.1 Introduction

As we stated earlier the plane problems of elasticity theory describe, approximately, the behavior of a thin elastic body (see Chapter 6.1). The plate theory is also an approximation to the three-dimensional problems of elasticity theory. The plate represents approximation of an elastic body when one dimension of the body is much smaller than other two. A plane dividing the thickness of the plate in half is called the middle plane. The stress tensor and displacement vector in plate theory are expressed as functions of points in the middle plane. As a matter of fact, this can be taken as a definition: a plate is an elastic body (no matter how thick) for which the stress tensor and displacement vector are functions of points on a single plane (called the middle plane) of a body.

Plates have wide application in engineering. We mention concrete and reinforced concrete plates in civil engineering, plates in naval architecture, in the automotive industry, and so on.

In this Chapter we present first a nonlinear theory of plates formulated by von Karman (1910). Then we also present a generalization of classical linear plate theory that includes the influence of shear stresses on the deformation of a plate. As far as loading is concerned we assume that the plate is loaded by an arbitrary system of forces having nonzero components in the direction normal to the middle plane as well as in the direction of the middle plane.

8.2 Basic equations of von Kármán's theory of plates

Consider a thin body (plate) as shown in Fig. 1. Suppose that the sides of the plate are of length $a$ and $b$ and that its thickness is $h$. We assume that $h << a$ and $h << b$. Let the axes of a rectangular Cartesian coordinate system $\bar{x}_i$ be oriented so that the $\bar{x}_1 - \bar{x}_2$ plane coincides with the middle plane of the plate. The origin is positioned at the corner of the plate so that the middle plane is defined as $0 \leq X_1 \leq a; 0 \leq X_2 \leq b$.

Suppose that the plate is loaded by a system of forces and couples on its edges and by uniformly distributed forces on its outer plane $X_3 = -h/2$. Let $u_i$ be the components of the displacement vector. The basic assumption of the von Kármán plate theory is that $u_1$ and $u_2$ are small with respect.
to \( u_3 \). Also the partial derivatives of \( u_1 \) and \( u_2 \) with respect to \( X_1 \) and \( X_2 \) are small when compared to partial derivatives of \( u_3 \) with respect to \( X_1 \) and \( X_2 \). From this assumption and \( x_1 = X_1 + u_1, x_2 = X_2 + u_2 \) it follows that the derivatives with respect to \( X_1 \) and \( X_2 \) could be replaced with the derivatives with respect to \( x_1 \) and \( x_2 \). Thus, the Lagrange-Green strain tensor, for a point on the middle plane, can be written as (see (2.2-16))

\[
\bar{E}_{11} = \frac{\partial u_1}{\partial x_1} + \frac{1}{2} \left( \frac{\partial u_3}{\partial x_1} \right)^2; \quad \bar{E}_{22} = \frac{\partial u_2}{\partial x_2} + \frac{1}{2} \left( \frac{\partial u_3}{\partial x_2} \right)^2; \\
\bar{E}_{12} = \frac{1}{2} \left( \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} + \frac{\partial u_3}{\partial x_1} \frac{\partial u_3}{\partial x_2} \right).
\]

(1)

In (1) we neglected all nonlinear terms but the ones that contain \( u_3 \) and its derivatives. From (1) it follows that

\[
\frac{\partial^2 \bar{E}_{11}}{\partial x_2^2} + \frac{\partial^2 \bar{E}_{22}}{\partial x_1^2} - 2 \frac{\partial^2 \bar{E}_{12}}{\partial x_1 \partial x_2} = \left( \frac{\partial^2 u_3}{\partial x_1 \partial x_2} \right)^2 - \frac{\partial^2 u_3}{\partial x_1^2} \frac{\partial^2 u_3}{\partial x_2^2}.
\]

(2)

The left-hand side of (2) is the compatibility equation for the linear strain tensor (see (2.7-13)). On the right-hand side we do not have a zero, since the \( \bar{E}_{ij} \) are components of the nonlinear strain tensor. Further analysis is based on the following assumptions (the Kirchhoff-Love hypothesis):

1. The line element normal to the middle plane before deformation transforms into a line element normal to the middle surface to which the middle plane is deformed. Also the length of the element before and after deformation remains the same.

2. Equidistant points from the middle plane do not influence each other; that is, there are no stress components in the direction normal to the middle plane.

The first hypothesis is equivalent to the Bernoulli-Mariotte hypothesis in rod theory. It has geometrical meaning. The second one could be expressed as

\[
\sigma_{33} = 0.
\]

(3)
It is obvious that the condition (3) is not satisfied exactly if the plate is loaded with the distributed forces normal to the middle plane.

The assumption that the plate is thin has the consequence that the components of the displacement vector $u_3$ for two arbitrary points $N$ and $M$ (see Fig. 2) are the same; that is,

$$u_{3N} = u_{3M} = W(x_1, x_2) . $$

By using (3) in Hooke’s law (see (6.1-19)) we obtain

$$
\sigma_{11} = \frac{E}{1 - \nu^2} (\bar{E}_{11} + \nu \bar{E}_{22}); \quad \sigma_{22} = \frac{E}{1 - \nu^2} (\bar{E}_{22} + \nu \bar{E}_{11});
$$

$$\sigma_{12} = \frac{E}{1 + \nu} \bar{E}_{12} . $$

From Fig. 3 it follows that the partial derivatives $\partial W/\partial x_1$ and $\partial W/\partial x_2$ define two angles $\theta_1$ and $\theta_2$ by the relations

$$
\tan \theta_1 = \frac{\partial W}{\partial x_1} \approx \theta_1; \quad \tan \theta_2 = \frac{\partial W}{\partial x_2} \approx \theta_2 .
$$

The components of the displacement vector for the point $N$ that is on the distance $x_3 \in [-h/2, h/2]$ from the middle plane can be obtained as a sum of the displacement vector for the point on the middle plane plus an additional
displacement vector. The components of this additional displacement vector are (see Fig. 3)$$\bar{u}_1 = -\theta_1 x_3; \quad \bar{u}_2 = -\theta_2 x_3. \quad (7)$$From (7) the components of the strain tensor at the point on the distance $x_3$ from the middle plane are

$$\tilde{E}^z_{11} = \frac{\partial u_1}{\partial x_1} + \frac{1}{2} \left( \frac{\partial W}{\partial x_1} \right)^2 - \frac{\partial^2 W}{\partial x_1^2} x_3; \quad \tilde{E}^z_{22} = \frac{\partial u_2}{\partial x_2} + \frac{1}{2} \left( \frac{\partial W}{\partial x_2} \right)^2 - \frac{\partial^2 W}{\partial x_2^2} x_3;$$

$$\tilde{E}^z_{12} = \frac{1}{2} \left( \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} + \frac{\partial W}{\partial x_1} \frac{\partial W}{\partial x_2} \right) - \frac{\partial^2 W}{\partial x_1 \partial x_2} x_3. \quad (8)$$

We now define cross-sectional quantities, contact forces, and contact couples. By using the notation from Fig. 4 we define

$$M_1 = \int_{-h/2}^{h/2} \sigma_{11} x_3 dx_3; \quad M_{12} = \int_{-h/2}^{h/2} \sigma_{12} x_3 dx_3;$$

$$M_2 = \int_{-h/2}^{h/2} \sigma_{22} x_3 dx_3; \quad T_1 = \int_{-h/2}^{h/2} \sigma_{11} x_3 dx_3;$$

$$T_2 = \int_{-h/2}^{h/2} \sigma_{22} x_3 dx_3; \quad S_{12} = \int_{-h/2}^{h/2} \sigma_{12} x_3 dx_3. \quad (9)$$

The quantities (9) define a contact force $F$ and a contact couple $M$ per unit length of the plate edge (in deformed or undeformed state, since $u_1$ and $u_2$ are small). The moments $M_1$ and $M_2$ are bending moments and $M_{12}$ is the twisting moment. By using (5) and (8) in (9) we obtain

$$M_1 = -D \left( \frac{\partial^2 W}{\partial x_1^2} + \nu \frac{\partial^2 W}{\partial x_2^2} \right); \quad T_1 = \frac{Eh}{1 - \nu^2} (\tilde{E}_{11} + \nu \tilde{E}_{22});$$

$$M_2 = -D \left( \frac{\partial^2 W}{\partial x_2^2} + \nu \frac{\partial^2 W}{\partial x_1^2} \right); \quad T_2 = \frac{Eh}{1 - \nu^2} (\tilde{E}_{22} + \nu \tilde{E}_{11});$$

$$M_{12} = -D (1 - \nu) \frac{\partial^2 W}{\partial x_1 \partial x_2}; \quad S_{12} = \frac{Eh}{1 + \nu} \tilde{E}_{12}, \quad (10)$$

where

$$D = \frac{Eh^3}{12(1 - \nu^2)}, \quad (11)$$

is the bending stiffness of the plate. From Fig. 4 we conclude that the quantities defined by (10) are positive when they have the direction shown in Fig. 5. In Fig. 5 we also showed the quantities (transversal forces)

$$Q_1 = \int_{-h/2}^{h/2} \sigma_{13} dx_3; \quad Q_2 = \int_{-h/2}^{h/2} \sigma_{23} dx_3. \quad (12)$$
The forces $Q_1$ and $Q_2$ are not constitutive quantities (since we did not assume the distribution of the $\sigma_{13}$ and $\sigma_{23}$ across the thickness of the plate). $Q_1$ and $Q_2$ must be determined from the equilibrium equations. It is possible to assume some distribution for the $\sigma_{13}$ and $\sigma_{23}$ (e.g., parabolic as in elementary rod theory) and then determine $Q_1$ and $Q_2$ by an expression of the type (9). However in this case we would violate the assumption 1 (i.e., the equivalent to the hypothesis plane section remains plane). The other possibility is to omit hypothesis 1. Then $Q_1$ and $Q_2$ become the constitutive quantities.
Since in the approximation taken here we have \( x_1 = X_1 \) and \( x_2 = X_2 \), it follows then that \( dX_1 = dx_1 \), \( dX_2 = dx_2 \). The sum of components of all forces in the \( \bar{x}_1 \) direction is

\[
\left( S_{12} + \frac{\partial S_{12}}{\partial x_2} dx_2 \right) dx_1 \cos \left( \frac{\partial W}{\partial x_1} + \frac{\partial^2 W}{\partial x_1 \partial x_2} dx_2 \right) - S_{12} dx_1 \cos \left( \frac{\partial W}{\partial x_1} \right)
\]

\[
+ \left( T_1 + \frac{\partial T_1}{\partial x_1} dx_1 \right) \cos \left( \frac{\partial W}{\partial x_1} + \frac{\partial^2 W}{\partial x_1 \partial x_2} dx_2 \right) dx_2 - T_1 \cos \left( \frac{\partial W}{\partial x_1} \right) dx_2
\]

\[
- Q_1 dx_2 \sin \left( \frac{\partial W}{\partial x_1} \right) - \left( Q_1 + \frac{\partial Q_1}{\partial x_1} dx_1 \right) dx_2 \sin \left( \frac{\partial^2 W}{\partial x_1^2} dx_1 + \frac{\partial W}{\partial x_1} \right)
\]

\[+ q_1 dx_1 dx_2 = 0. \tag{13}\]

Similarly we can write the sum of forces in \( \bar{x}_2 \) directions. By dividing by \( dx_1 dx_2 \) and by neglecting the higher-order quantities, we obtain

\[
\frac{\partial T_1}{\partial x_1} + \frac{\partial S_{12}}{\partial x_2} - Q_1 \frac{\partial^2 W}{\partial x_1^2} - \frac{\partial W}{\partial x_1} \frac{\partial Q_1}{\partial x_1} + q_1 = 0;
\]

\[
\frac{\partial T_2}{\partial x_2} + \frac{\partial S_{12}}{\partial x_1} - Q_2 \frac{\partial^2 W}{\partial x_2^2} - \frac{\partial W}{\partial x_2} \frac{\partial Q_2}{\partial x_2} + q_2 = 0, \tag{14}\]

where \( q_1 \) and \( q_2 \) are components of the distributed forces in the \( \bar{x}_1 \) and \( \bar{x}_2 \) directions.

We determine next the sum of projections of all forces in the \( \bar{x}_3 \) direction. Note that in the deformed state the angle between the forces \( T_1 dx_2 \) and \( [T_1 + (\partial T_1/\partial x_1) dx_1] dx_2 \) is \( (\partial^2 W/\partial x_1^2) dx_1 \). The projection of those two forces on the \( \bar{x}_3 \) direction, up to and including second-order terms, is given by \( T_1(\partial^2 W/\partial x_1^2) dx_1 dx_2 + (\partial T_1/\partial x_1)(\partial W/\partial x_1) dx_1 dx_2 \). Similarly the forces \( T_2 dx_1 \) and \( [T_2 + (\partial T_2/\partial x_2) dx_2] dx_1 \) have the projection \( T_2(\partial^2 W/\partial x_2^2) dx_1 dx_2 + (\partial T_2/\partial x_2)(\partial W/\partial x_2) dx_1 dx_2 \) on the \( \bar{x}_3 \) axis. The angle between \( S_{12} dx_1 \) and \( S_{12}(\partial^2 W/\partial x_1 \partial x_2) dx_1 dx_2 \) is \( (\partial^2 W/\partial x_1 \partial x_2) dx_2 \) so that their joint projection on the \( \bar{x}_3 \) axis is \( S_{12}(\partial^2 W/\partial x_1 \partial x_2) dx_1 dx_2 + (\partial S_{12}/\partial x_1)(\partial W/\partial x_2) dx_1 dx_2 \). The same joint projection has forces \( S_{12} \) that act on the sides of length \( dx_2 \). Finally \( Q_1 dx_1 \) and \( [Q_1 + (\partial Q_1/\partial x_2) dx_2] dx_1 \) have the projection given by \( (\partial Q_1/\partial x_1) dx_1 dx_2 \) and the forces \( Q_2 dx_1 \) and \( [Q_2 + (\partial Q_2/\partial x_2) dx_2] dx_1 \) have the projection \( (\partial Q_2/\partial x_2) dx_1 dx_2 \). By adding to those forces the projection of the distributed forces and by neglecting the higher-order terms, we obtain

\[
\frac{\partial Q_1}{\partial x_1} + \frac{\partial Q_2}{\partial x_2} + T_1 \frac{\partial^2 W}{\partial x_1^2} + T_2 \frac{\partial^2 W}{\partial x_2^2} + 2S_{12} \frac{\partial^2 W}{\partial X_1 \partial X_2} + q_1 \frac{\partial W}{\partial x_1} + q_2 \frac{\partial W}{\partial x_2} + q_3 = 0. \tag{15}\]
The sum of all moments leads to
\[
\frac{\partial M_1}{\partial x_1} + \frac{\partial M_{12}}{\partial x_2} - Q_1 = 0; \quad \frac{\partial M_2}{\partial x_2} + \frac{\partial M_{12}}{\partial x_1} - Q_2 = 0. \tag{16}
\]

By writing (16) we used the sign convention shown in Fig. 5.

The system of equations is now complete. Namely, the equilibrium equations (14), (15), (16) together with constitutive equations (10) and geometrical condition (1) could be used to determine: the components of forces \(T_1, T_2, S_{12}, Q_1, Q_2\) and couples \(M_1, M_2, M_{12}\), strain tensor components \(\bar{E}_{11}, \bar{E}_{22}, \bar{E}_{12}\), and displacement vector components \(u_1, u_2, W\). The strain
tensor components must satisfy the condition (2).

We proceed to solve the simplified system of equations. Suppose that \(q_1 = q_2 = 0\). Furthermore, we assume that \(Q_1(\partial^2 W/\partial x_1^2), (\partial Q_1/\partial x_1)(\partial W/\partial x_1), Q_2(\partial^2 W/\partial x_2^2)\), and \((\partial Q_1/\partial x_1)(\partial W/\partial x_1)\) are so small we can neglect those terms. Then (14) becomes
\[
\frac{\partial T_1}{\partial x_1} + \frac{\partial S_{12}}{\partial x_2} = 0; \quad \frac{\partial T_2}{\partial x_2} + \frac{\partial S_{12}}{\partial x_1} = 0. \tag{17}
\]

By introducing the “stress function” \(\Phi(x_1, x_2)\) by the relation
\[
T_1 = \frac{\partial^2 \Phi}{\partial x_2^2}; \quad T_2 = \frac{\partial^2 \Phi}{\partial x_1^2}; \quad S_{12} = -\frac{\partial^2 \Phi}{\partial x_1 \partial x_2}, \tag{18}
\]
the conditions (17) are identically satisfied. Next we solve (10) for \(\bar{E}_{ij}\) and use the result in (18). Thus, we obtain
\[
\bar{E}_{11} = \frac{1}{Eh} (T_1 - \nu T_2) = \frac{1}{Eh} \left( \frac{\partial^2 \Phi}{\partial x_2^2} - \nu \frac{\partial^2 \Phi}{\partial x_1^2} \right);
\]
\[
\bar{E}_{22} = \frac{1}{Eh} (T_2 - \nu T_1) = \frac{1}{Eh} \left( \frac{\partial^2 \Phi}{\partial x_1^2} - \nu \frac{\partial^2 \Phi}{\partial x_2^2} \right);
\]
\[
\bar{E}_{12} = \frac{1 + \nu}{Eh} S_{12} = -\frac{1 + \nu}{Eh} \frac{\partial^2 \Phi}{\partial x_1 \partial x_2}. \tag{19}
\]

With (19) equation (2) becomes
\[
\frac{1}{Eh} \nabla^4 \Phi = \left( \frac{\partial^2 W}{\partial x_1 \partial x_2} \right)^2 - \frac{\partial^2 W}{\partial x_1^2} \frac{\partial^2 W}{\partial x_2^2}. \tag{20}
\]

It remains to satisfy equations (15) and (16). If we determine \(Q_1\) and \(Q_2\) in terms of the derivatives of \(M_1, M_2, \) and \(M_{12}\) and in the result use (10), we obtain
\[
Q_1 = -D \left( \frac{\partial^3 W}{\partial x_1^3} + \frac{\partial^3 W}{\partial x_1 \partial x_2^2} \right); \quad Q_2 = -D \left( \frac{\partial^3 W}{\partial x_2^3} + \frac{\partial^3 W}{\partial x_1^2 \partial x_2} \right). \tag{21}
\]
Finally by substituting (21) and (18) in (15) we obtain

\[ D\nabla^4 W = q_3 + \frac{\partial^2 \Phi}{\partial x_2^2} \frac{\partial^2 W}{\partial x_1^2} + \frac{\partial^2 \Phi}{\partial x_1^2} \frac{\partial^2 W}{\partial x_2^2} - 2 \frac{\partial^2 \Phi}{\partial x_1 \partial x_2} \frac{\partial^2 W}{\partial x_1 \partial x_2}. \]  

Equations (20) and (22) are the von Kármán equations. They were derived by von Kármán (1910). The derivation of the equations from the three-dimensional elasticity theory by an asymptotic method as well as the proof of the existence of solution analysis is presented by Ciarlet and Rabier (1980).

In equations (20),(22) the biharmonic operator is

\[ \nabla^4 (\cdot) = \frac{\partial^4}{\partial x_1^4} (\cdot) + 2 \frac{\partial^4}{\partial x_1^2 \partial x_2^2} (\cdot) + \frac{\partial^4}{\partial x_2^4} (\cdot). \]  

Equation (24) is the one obtained by Lagrange and Sophie Germain in 1811 (see Timoshenko and Woinowsky-Krieger (1959)).

8.3 Boundary conditions

The boundary conditions corresponding to the system (8.2-20),(8.2-22) are formulated as the following.

1. **The built-in end**

   If the edge \( x_1 = a \) is built in we have

   \[ W(x_1 = a, x_2) = 0; \quad \frac{\partial W}{\partial x_1}(x_1 = a, x_2) = 0. \]  

2. **Freely supported end**

   When the edge is freely supported we have that the vertical displacement \( W \) and the moment \( M_1 \) must be equal to zero. Thus,

   \[ W(x_1 = a, x_2) = 0; \quad \left( \frac{\partial^2 W}{\partial x_1^2} + \nu \frac{\partial^2 W}{\partial x_2^2} \right)_{x_1 = a} = 0, \]  

where we used (8.2-10). However the condition \( W = 0 \) on the edge \( x_1 = a \) implies \( \partial^2 W/\partial x_2^2 = 0 \). Then, (2) becomes

   \[ W(x_1 = a, x_2) = 0; \quad \left. \frac{\partial^2 W}{\partial x_1^2} \right|_{x_1 = a} = 0. \]  

Examples of built-in and freely supported plates are shown in Fig. 7.
We show that the conditions $M_{12} = 0$ and $Q_1 = 0$ are not independent. Thus we consider a part of the edge $x_1 = 0$. Let the length of this part be $2dx_2$ (see Fig. 8).

The couple that acts on the part of the edge of length $dx_2$ is $M_{12}dx_2$. This couple can be replaced by two concentrated forces of intensity $M_{12}$. The action lines of those forces are $dx_2$ apart. For the neighboring element of the length $dx_2$ the couple is $(M_{12} + dM_{12})dx_2$. This couple too can be replaced by two concentrated forces of intensity $(M_{12} + dM_{12})$ with action lines $dx_2$ apart. In Fig. 8 we showed the vectors of these forces by dashed lines. By adding the forces $M_{12}$ and $(M_{12} + dM_{12})$ we conclude that the $dM_{12}$ can be represented (is statically equivalent) to the force

$$\bar{Q}_1 = \frac{\partial M_{12}}{\partial x_2}.$$  

(5)
According to the Saint-Venant principle the fact that we replaced $M_{12}$ by $\bar{Q}_1$ has only local effect in the vicinity of the edge $x_1 = a$. The conditions $(4)_{2,3}$ could now be written as

\[(Q_1 + \bar{Q}_1)_{x_1=a} = 0,\]  

or

\[\left(Q_1 + \frac{\partial M_{12}}{\partial x_2}\right)_{x_1=a} = 0.\]  

Finally by using (8.1-10) and (8.1-21) the conditions (4) and (7) become

\[
\begin{align*}
\left[\frac{\partial^3 W}{\partial x_1^3} + (2 - \nu)\frac{\partial^3 W}{\partial x_1 \partial x_2^2}\right]_{x_1=a} &= 0; \\
\left[\frac{\partial^2 W}{\partial x_1^2} + \nu \frac{\partial^2 W}{\partial x_2^2}\right]_{x_1=a} &= 0.
\end{align*}
\]  

The conditions (8) correspond to the free edge of the plate. In the form (8) they were obtained by Kirchhoff in 1850.

### 8.4 Small deformations: An example

Consider a rectangular plate with sides of length $a$ and $b$. Suppose that the plate is loaded by a uniformly distributed load of the form

\[q_1 = 0; \quad q_2 = 0; \quad q_3 = q_0 \sin(\pi x_1/a) \sin(\pi x_2/b),\]  

where $q_0$ is a constant. In the study of small deformations the equation relevant for determining $W$ is (8.2-24). Therefore

\[\frac{\partial^4 W}{\partial x_1^4} + 2\frac{\partial^4 W}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4 W}{\partial x_2^4} = \frac{q_0}{D} \sin(\pi x_1/a) \sin(\pi x_2/b).\]  

The boundary conditions for the simply supported plate are

\[W = 0, \quad \frac{\partial^2 W}{\partial x_1^2} = 0, \quad \text{for} \quad x_1 = 0 \quad \text{and} \quad x_1 = a;\]

\[W = 0, \quad \frac{\partial^2 W}{\partial x_2^2} = 0, \quad \text{for} \quad x_2 = 0, \quad \text{and} \quad x_2 = b.\]  

We assume $W$ in the form

\[W = C \sin(\pi x_1/a) \sin(\pi x_2/b),\]  

where $C$ is a constant. By substituting (4) in (2) we obtain

\[\pi^4 \left(\frac{1}{a^2} + \frac{1}{b^2}\right)^2 C = \frac{q_0}{D}.\]
Solving for \( C \) and substituting the result in (4) we obtain

\[
W = \frac{q_0}{\pi^4 D \left( \frac{1}{a^2} + \frac{1}{b^2} \right)^2} \sin \frac{\pi x_1}{a} \sin \frac{\pi x_2}{b} .
\] (6)

The solution (6) could be used to obtain the solution for a simply supported plate with more general loading. Suppose that the loading \( q_3 \) is given by

\[
q_3 = q_3(x_1, x_2).
\] (7)

We first expand the function \( q_3 \) in a Fourier series as

\[
q_3 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} \sin \left( \frac{m\pi x_1}{a} \right) \sin \left( \frac{n\pi x_2}{b} \right) .
\] (8)

The coefficients \( a_{mn} \) we determine as follows. First, multiply (8) by

\[
\sin \frac{k\pi x_2}{b} ,
\] (9)

and integrate with respect to \( x_2 \) between 0 and \( b \). The result is

\[
\int_0^b q_3(x_1, x_2) \sin \left( \frac{k\pi x_2}{b} \right) \, dx_2 = \frac{b}{2} \sum_{m=1}^{\infty} a_{mk} \sin \left( \frac{m\pi x_1}{b} \right) .
\] (10)

In writing (10) we used

\[
\int_0^b \sin \left( \frac{n\pi x_2}{b} \right) \sin \left( \frac{k\pi x_2}{b} \right) \, dx_2 = \begin{cases} 0 & n \neq k \\ b/2 & n = k \end{cases} .
\] (11)

Next we multiply (10) by

\[
\sin \frac{m\pi x_1}{a} ,
\] (12)

and integrate with respect to \( x_1 \) between 0 and \( a \). The result is

\[
\int_0^a \int_0^b q_3(x_1, x_2) \sin \left( \frac{m\pi x_1}{a} \right) \sin \left( \frac{k\pi x_2}{b} \right) \, dx_1 \, dx_2 = \frac{ab}{4} a_{km} .
\] (13)

From (13) it follows that

\[
a_{km} = \frac{4}{ab} \int_0^1 \int_0^b q_3(x_1, x_2) \sin \left( \frac{m\pi x_1}{a} \right) \sin \left( \frac{k\pi x_2}{b} \right) \, dx_1 \, dx_2 .
\] (14)

Since the load (8) is the sum of the loads of the type (1) and since (2) is a linear differential equation, it follows that the solution of (8.2-24) with \( q_3 \) given by (8) is

\[
W = \frac{1}{\pi^4 D} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_{mn}}{\left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2} \sin \frac{m\pi x_1}{a} \sin \frac{n\pi x_2}{b} ,
\] (15)

where \( a_{mn} \) is given by (14).
8.5 The influence of shear stresses: Reissner-Mindlin theory

We present a generalization of the plate theory proposed by Reissner (1945) and Mindlin (1951). The theory is based on slightly changed assumptions from those stated in Section 8.2. Namely, here we introduce the following assumption (compare with assumption 1 of Section 8.2).

1. The line element normal to the middle plane before deformation transforms into a line element of the same length but not necessarily normal to the middle surface.

Thus, in the Reissner-Mindlin theory the rotation of a line element that in the undeformed state is orthogonal to the middle plane is not determined by the shape of the middle plane in the deformed state. In classical theory we used (see (8.2-6), (8.2-7) \( \partial W / \partial x_1 = \theta_1, \partial W / \partial x_2 = \theta_2 \)). Here we assume, instead of (8.2-3), (8.2-7)

\[
\begin{align*}
\bar{u}_1 &= \bar{\theta}_1 x_3; & \bar{u}_2 &= \bar{\theta}_2 x_3; & \bar{u}_3 &= W(x_1, x_2),
\end{align*}
\]

where \( \bar{\theta}_1 \) and \( \bar{\theta}_2 \) are the angles of rotation of the line element (originally orthogonal to the middle plane) that must be determined. In what follows we assume that there is no deformation of the plate's middle plane so that the \( \bar{E}_{ij} \) given by (8.2-1) are equal to zero. The strain tensor for a point on the distance \( x_3 \) from the middle plane becomes

\[
\begin{align*}
E_{11}^z &= \frac{\partial \bar{\theta}_1}{\partial x_1} x_3; & E_{22}^z &= \frac{\partial \bar{\theta}_2}{\partial x_1} x_3; & E_{33}^z &= 0; & E_{12}^z &= \frac{1}{2} \left( \frac{\partial \bar{\theta}_1}{\partial x_2} + \frac{\partial \bar{\theta}_2}{\partial x_1} \right) x_3; \\
E_{23}^z &= \frac{1}{2} \left( \frac{\partial W}{\partial x_2} \right); & E_{13}^z &= \frac{1}{2} \left( \frac{\partial \bar{\theta}_1}{\partial x_1} + \frac{\partial W}{\partial x_1} \right).
\end{align*}
\]

Note that now (2) represents the total deformation tensor (that is the reason why we omitted the bar over \( E_{ij} \)). By using Hooke's law (6.1-19) and (2) we obtain

\[
\begin{align*}
\sigma_{11} &= \frac{E}{1 - \nu^2} [E_{11} + \nu E_{22}] = \frac{E}{1 - \nu^2} \left[ \frac{\partial \bar{\theta}_1}{\partial x_1} + \nu \frac{\partial \bar{\theta}_2}{\partial x_2} \right] x_3; \\
\sigma_{22} &= \frac{E}{1 - \nu^2} [E_{22} + \nu E_{11}] = \frac{E}{1 - \nu^2} \left[ \frac{\partial \bar{\theta}_2}{\partial x_2} + \nu \frac{\partial \bar{\theta}_1}{\partial x_1} \right] x_3; \\
\sigma_{12} &= \frac{E}{2(1 + \nu)} \left[ \frac{\partial \bar{\theta}_1}{\partial x_2} + \frac{\partial \bar{\theta}_2}{\partial x_1} \right] x_3; \\
\sigma_{13} &= \frac{E}{2(1 + \nu)} \left[ \frac{\partial \bar{\theta}_1}{\partial x_1} + \frac{\partial W}{\partial x_1} \right]; & \sigma_{23} &= \frac{E}{2(1 + \nu)} \left[ \frac{\partial \bar{\theta}_2}{\partial x_2} + \frac{\partial W}{\partial x_2} \right].
\end{align*}
\]
8.5 The influence of shear stresses: Reissner-Mindlin theory

With (3) we can calculate the cross-sectional quantities defined by (8.2-9) and (8.2-12). Thus, we obtain

$$M_1 = D \left( \frac{\partial \bar{\theta}_1}{\partial x_1} + \nu \frac{\partial \bar{\theta}_2}{\partial x_2} \right); \quad M_2 = D \left( \frac{\partial \bar{\theta}_2}{\partial x_2} + \nu \frac{\partial \bar{\theta}_1}{\partial x_1} \right);$$

$$M_{12} = D \frac{1 - \nu}{2} \left( \frac{\partial \bar{\theta}_1}{\partial x_2} + \frac{\partial \bar{\theta}_2}{\partial x_1} \right); \quad T_1 = T_2 = S_{12} = 0,$$

where $D$ is given by (8.2-11). To determine $Q_1$ and $Q_2$ according to (8.2-12) we need the distribution of $\sigma_{13}$ and $\sigma_{23}$ across the thickness of the plate. According to (3) those stress components are independent of $x_3$. In reality this is not so since the condition $\sigma_{13} = \text{const.}$ and $\sigma_{23} = \text{const.}$ are not satisfied for $x_3 = \pm h/2$ (as a matter of fact, the planes $x_3 = \pm h/2$ are stress free). For this reason we introduce the shear thickness $h_s < h$ of the plate. Then, by using (3) in (8.2-12) we obtain

$$Q_1 = \frac{E}{2(1 + \nu)} h_s \left( \bar{\theta}_1 + \frac{\partial W}{\partial x_1} \right); \quad Q_2 = \frac{E}{2(1 + \nu)} h_s \left( \bar{\theta}_2 + \frac{\partial W}{\partial x_2} \right).$$

The quantity $h_s$ takes into account the nonuniformity of shear stresses. Often it is written in the form $h_s = kh$ where $k$ is the correction factor. Its value was given by Reissner ($k = 5/6$) and by Mindlin ($k = \pi^2/12$).

Further procedure is similar to that of Section 8.2. Namely, in the present case the equilibrium equations (8.2-15), (8.2-16) become

$$\frac{\partial Q_1}{\partial x_1} + \frac{\partial Q_2}{\partial x_2} + q_3 = 0; \quad \frac{\partial M_1}{\partial x_1} + \frac{\partial M_{12}}{\partial x_2} - Q_1 = 0;$$

$$\frac{\partial M_2}{\partial x_2} + \frac{\partial M_{12}}{\partial x_1} - Q_2 = 0,$$

where we used $T_1 = T_2 = S_{12} = 0$. From (4) through (6) we can, in principle, determine the following eight quantities: five cross-sectional quantities ($M_{11}$, $M_{22}$, $M_{12}$, $Q_1$ and $Q_2$) and three generalized displacements ($\bar{\theta}_1$, $\bar{\theta}_2$, and $W$). We can simplify the system (4) through (6). For example, from (6) we have

$$\frac{\partial^2 M_1}{\partial x_1^2} + 2 \frac{\partial^2 M_{12}}{\partial x_1 \partial x_2} + \frac{\partial^2 M_2}{\partial x_2^2} = -q_3.$$

Let $\Phi$ and $\Psi$ be functions defined by (Marguerre and Woernle 1975)

$$\Phi = \frac{\partial \bar{\theta}_1}{\partial x_1} + \frac{\partial \bar{\theta}_2}{\partial x_2}; \quad \Psi = \frac{\partial \bar{\theta}_2}{\partial x_1} - \frac{\partial \bar{\theta}_1}{\partial x_2}.$$

Then (7) after the use of (4) and (8) becomes

$$DV^2 \Phi = -q_3.$$
Also by using (4) in (6) and solving the resulting system for derivatives of \( W \), we obtain

\[
\frac{\partial W}{\partial x_1} = -\bar{\theta}_1 + \frac{D}{\mu h_s} \left( \frac{\partial \Phi}{\partial x_1} - \frac{1 - \nu}{2} \frac{\partial \Psi}{\partial x_2} \right);
\]

\[
\frac{\partial W}{\partial x_2} = -\bar{\theta}_2 + \frac{D}{\mu h_s} \left( \frac{\partial \Phi}{\partial x_2} - \frac{1 - \nu}{2} \frac{\partial \Psi}{\partial x_1} \right),
\]

where \( \mu = E/2(1 + \nu) \) is the Lamé constant (shear modulus). By differentiating (10)_1 with respect to \( x_1 \) and (10)_2 with respect to \( x_2 \) and by adding the results, it follows that

\[
\nabla^2 W = -\Phi + \frac{D}{\mu h_s} \nabla^2 \Phi.
\]

Similarly by differentiating (10)_1 with respect to \( x_2 \) and (10)_2 with respect to \( x_1 \) and by subtracting the results, we obtain

\[
\Psi - \frac{D}{\mu h_s} \frac{1 - \nu}{2} \nabla^2 \Psi = 0.
\]

Therefore, the Reissner-Mindlin plate theory is described by the following system of equations

\[
D\nabla^2 \Phi = -q_3; \quad \nabla^2 W = -\Phi + \frac{D}{\mu h_s} \nabla^2 \Phi;
\]

\[
\Psi - \frac{D}{\mu h_s} \frac{1 - \nu}{2} \nabla^2 \Psi = 0.
\]

Note that for the case when shear modulus \( \mu \) tends to infinity, we obtain from (13)

\[
D\nabla^4 \Phi = -q_3; \quad \nabla^2 W = -\Phi; \quad \Psi = 0.
\]

We turn now to the boundary conditions for the Reissner-Mindlin plates. Those conditions correspond to the system (13) and read:

1. The built-in end

If the edge \( x_1 = a \) is built-in, we have that the displacement and angles of rotation are equal to zero. This is the so-called hard condition

\[
W(x_1 = a, x_2) = 0; \quad \bar{\theta}_1(x_1 = a, x_2) = 0; \quad \bar{\theta}_2(x_1 = a, x_2) = 0.
\]

Another way to specify boundary conditions is the so-called soft condition that reads

\[
W(x_1 = a, x_2) = 0; \quad M_{12}(x_1 = a, x_2) = 0; \quad \bar{\theta}_2(x_1 = a, x_2) = 0.
\]
2. Free end

If the edge \( x_1 = a \) is free, then

\[
M_1(x_1 = a, x_2) = 0; \quad M_{12}(x_1 = a, x_2) = 0; \quad Q_1(x_1 = a, x_2) = 0. \tag{17}
\]

In conditions (15) through (17) equations (4) and (10) should be used.

We conclude the analysis of the Reissner-Mindlin plate theory by the following observations.

1. For the case when the shear rigidity of the plate is large (i.e., \( \mu h_s \to \infty \)) we obtain from (10)

\[
\frac{\partial W}{\partial x_1} = -\bar{\theta}_1; \quad \frac{\partial W}{\partial x_2} = -\bar{\theta}_2. \tag{18}
\]

Note that (18) is in agreement with (8.2-7) and (8.5-1). Also equations (14) reduce to

\[
D \nabla^4 W = q_3. \tag{19}
\]

Thus, the Reissner-Mindlin theory reduces to the case (8.2-24).

2. Since the system (13) is the system of three partial differential equations of the second-order on each edge we can prescribe three boundary conditions. In classical theory the relevant equation was (8.2-24), that is, a fourth-order partial differential equation. Thus on each edge it was possible to prescribe only two boundary conditions. For the case of a free edge (see Section 8.3) we had to replace the derivative of the couple \( M_{12} \) with the transversal force \( (Q_1 \text{ in (8.3-5)}) \). This was possible since the transversal forces cause no "shear effect" (\( \mu h_s \to \infty \)). Here such replacement is not needed.

We now present a concrete example for the Reissner-Mindlin theory. Consider a plate \( 0 \leq x_1 \leq a, \quad 0 \leq x_2 \leq b \) with \( a \ll b \) as shown in Fig. 9. We assume that the plate is simply supported at all sides and loaded by distributed couples along the side \( x_2 = 0 \).
The system (13) now becomes
\[ D\nabla^2 \Phi = 0; \quad \nabla^2 W = -\Phi; \quad \Psi - \frac{1}{\kappa_s} \nabla^2 \Psi = 0, \] (20)
where
\[ \frac{1}{\kappa_s} = \frac{D}{\mu h_s} \left( 1 - \nu \right) \] (21).
Suppose that the solution of (20) is taken as
\[ \Phi = \sum_{n=1}^{\infty} f_n(x_2) \sin \alpha_n x_1; \quad W = \sum_{n=1}^{\infty} h_n(x_2) \sin \alpha_n x_1; \]
\[ \Psi = \sum_{n=1}^{\infty} g_n(x_2) \sin \alpha_n x_1, \] (22)
where \( \alpha_n = n\pi / a \). Also in (22) the functions \( f_n, h_n, g_n \) are to be determined. By substituting (22) into (20) we obtain
\[ D \left( -\alpha_n^2 f_n + \frac{d^2 f_n}{dx_2^2} \right) = 0; \quad -\alpha_n^2 h_n + \frac{d^2 h_n}{dx_2^2} = f_n; \]
\[ g_n - \frac{1}{\kappa_s} \left( -\alpha_n^2 g_n + \frac{d^2 g_n}{dx_2^2} \right) = 0. \] (23)
The solution of (23) that decreases with \( x_2 \) is
\[ f_n = C_n \exp(-\alpha_n x_2); \quad h_n = \left( A_n + \frac{C_n}{2\alpha_n} \right) \exp(-\alpha_n x_2); \]
\[ g_n = D_n \exp(-\lambda_n x_2), \] (24)
where
\[ \lambda_n^2 = \kappa_s + \alpha_n^2. \] (25)
By using (10) and (22) in (4) we get
\[ M_{22} = D \left[ \left( \frac{\partial^2 \Theta_2}{\partial x_2^2} + \nu \left( \frac{\partial \Theta_1}{\partial x_1} \right) \right) \right] = D \left[ -\left( \frac{\partial^2 W}{\partial x_2^2} \right) - \nu \left( \frac{\partial^2 W}{\partial x_1^2} \right) \right] \]
\[ + \frac{D}{\mu h_s} \left\{ \left( \frac{\partial^2 \Phi}{\partial x_2^2} \right) + \left( \nu \frac{\partial^2 \Phi}{\partial x_1^2} \right) \right\} + \frac{1}{\kappa_s} \left\{ \left( \frac{\partial^2 \Psi}{\partial x_1 \partial x_2} \right) - \nu \left( \frac{\partial^2 \Psi}{\partial x_1 \partial x_2} \right) \right\} \]
\[ = D \left\{ -\alpha_n^2 A_n + \nu \alpha_n^2 A_n + \frac{C_n}{2\alpha_n} \left( 2\alpha_n - \alpha_n^2 x_2 + \nu \alpha_n^2 x_2 \right) \right\} \exp(-\alpha_n x_2) \]
\[ + \frac{D}{\mu h_s} \left( \alpha_n^2 - \nu \alpha_n^2 \right) C_n \exp(-\alpha_n x_2) + \frac{1}{\kappa_s} \lambda_n \alpha_n (1 - \nu) D_n \exp(-\lambda_n x_2) \sin \alpha_n x_1; \]
The influence of shear stresses: Reisner-Mindlin theory

\[ M_{12} = \frac{1 - \nu}{2} D \left[ \left( \frac{\partial^2}{\partial x_1^2} \right) + \left( \frac{\partial^2}{\partial x_2^2} \right) \right] = \frac{1 - \nu}{2} D \left[ \frac{\partial^2 W}{\partial x_1 \partial x_2} \right] \]

\[ + \frac{D}{\mu h_s} \left\{ 2 \left( \frac{\partial^2 \Phi}{\partial x_1 \partial x_2} \right) \right\} + \frac{1 - \nu}{2} \left\{ \left( \frac{\partial^2 \Psi}{\partial x_1^2} \right) - \left( \frac{\partial^2 \Psi}{\partial x_2^2} \right) \right\} \]

\[ = \frac{1 - \nu}{2} D \left\{ 2\alpha_n^2 A_n + \frac{C_n}{2\alpha_n} (\alpha_n - \alpha_n^2) \right\} \exp(-\alpha_n x_2) \]

\[ + \frac{D}{\mu h_s} (-2\alpha_n^2) C_n \exp(-\alpha_n x_2) + \frac{1 - \nu}{2} \left( -\alpha_n^2 - \lambda_n^2 \right) D_n \exp(-\lambda_n x_2) \cos \alpha_n x_1. \quad (26) \]

Next we expand the distributed couples \( M_0 \) in a Fourier sine series as

\[ M_0 = \sum_{n=1}^{\infty} M_n \sin \alpha_n x_1. \quad (27) \]

The boundary conditions for the plate shown in Fig. 9 are

\[ W(x_1, x_2 = 0) = 0; \quad M_{22}(x_1, x_2 = 0) = M_0; \quad M_{12}(x_1, x_2 = 0) = 0. \quad (28) \]

By using (26),(27) in (28) we obtain

\[ A_n = 0; \quad \frac{M_n}{D} = C_n \left[ 1 + \frac{D}{\mu h_s} \alpha_n^2 (1 - \nu) \right] + \frac{1}{\kappa_s} \lambda_n \alpha_n (1 - \nu) D_n; \]

\[ \frac{C_n}{2} + \frac{D}{\mu h_s} \left[ -2\alpha_n^2 C_n + \frac{1 - \nu}{2} (-\alpha_n^2 - \lambda_n^2) D_n \right] = 0. \quad (29) \]

In Eschenauer and Schnell (1986) the constants \( C_n \) and \( D_n \) are determined in the special case when

\[ h_s = \frac{5}{6}; \quad \frac{1}{\kappa_s} = \frac{1 - \nu}{2} \frac{Eh^3}{12(1 - \nu^2)} \frac{2(1 + \nu)}{E} \approx h^2 \frac{6}{5h} \approx \frac{h^2}{10}. \quad (30) \]

Then

\[ C_n = \frac{M_n}{D} \frac{\alpha_n^2}{\kappa_s} - \frac{1}{1 + 2\alpha_n \kappa_s + \frac{1}{\alpha_n^2 + \lambda_n^2} \left( (1 - \nu) \alpha_n \lambda_n + \frac{2}{3} h^2 \alpha_n^3 \lambda_n \right)} = \frac{M_n}{D} \Lambda_n. \quad (31) \]

With (29) and (31) we obtain from (22)

\[ W = \sum_{n=1}^{\infty} \frac{M_n}{2\alpha_n} \left[ \Lambda_n \alpha_n x_2 \exp(-\alpha_n x_2) \right] \sin \frac{n\pi}{a} x_1. \quad (32) \]

By analyzing the numerical value of \( W \) in one specific case it was shown in Eschenauer and Schnell (1986) that finite shear rigidity increases \( W \) when compared with the classical theory described with (8.2-24).
Problems

1. Show that for the case of axially symmetric deformation, that is, $W(r, \theta) = W(r)$, $\Phi(r, \theta) = \Phi(r)$, the equations of von Kármán’s theory (8.2-20), (8.2-22) reduce to

$$\frac{1}{Eh} \frac{1}{r} \frac{d}{dr} \left\{ r \frac{d}{dr} \left[ \frac{1}{r} \frac{d}{dr} \left( r \frac{dW}{dr} \right) \right] \right\} = - \frac{1}{2r} \frac{d}{dr} \left( \frac{dW}{dr} \right)^2;$$

$$D \frac{1}{r} \frac{d}{dr} \left\{ r \frac{d}{dr} \left[ \frac{1}{r} \frac{d}{dr} \left( r \frac{dW}{dr} \right) \right] \right\} = q_3 + \frac{d^2\Phi}{dr^2} \frac{1}{r} \frac{dW}{dr} + \frac{1}{r} \frac{d\Phi}{dr} \frac{d^2W}{dr^2}.$$

2. For the case of axially symmetric deformation of the previous example, show that

$$T_r = \frac{1}{r} \frac{d\Phi}{dr}; \quad T_\theta = \frac{d^2\Phi}{dr^2}. \quad (a)$$

Then by using

$$T_\theta = \frac{d}{dr} (rT_r), \quad (b)$$

show that the first equation of Problem 1 could be integrated so that the following system is obtained

$$\frac{1}{Eh} \frac{1}{r} \frac{d}{dr} \left\{ r \frac{d}{dr} \left( r^2 T_r \right) \right\} = - \frac{1}{2} \left( \frac{dW}{dr} \right)^2;$$

$$D \frac{1}{r} \frac{d}{dr} \left\{ r \frac{d}{dr} \left[ \frac{1}{r} \frac{d}{dr} \left( r \frac{dW}{dr} \right) \right] \right\} = q_3 + \frac{1}{r} \frac{d}{dr} \left( rT_r \frac{dW}{dr} \right). \quad (c)$$

Finally show that from the condition (8.2-19) written in the cylindrical coordinate system we have

$$\bar{E}_{\theta\theta} = \frac{1}{Eh} (T_\theta - \nu T_r), \quad (d)$$

which together with (2.6-6) leads to the expression for the radial component of the displacement vector as

$$u = u_r = \frac{r}{Eh} (T_\theta - \nu T_r) = \frac{r}{Eh} \left[ \frac{d}{dr} (rT_r) - \nu T_r \right]. \quad (e)$$

3. Show that in the cylindrical coordinate system equation (8.2-24) becomes

$$\left( \frac{\partial^4\Phi}{\partial r^4} + \frac{2}{r} \frac{\partial^3\Phi}{\partial r^3} - \frac{1}{r^2} \frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r^3} \frac{\partial\Phi}{\partial r} \right)$$

$$+ \left( \frac{2}{r^2} \frac{\partial^4\Phi}{\partial r^2 \partial \theta^2} - \frac{2}{r^3} \frac{\partial^3\Phi}{\partial r \partial \theta^2} \right) + \frac{1}{r^4} \left( \frac{\partial^4\Phi}{\partial^4 \theta^4} + \frac{4}{\partial \theta^2} \frac{\partial^2\Phi}{\partial \theta^2} \right) = \frac{q_3}{D}. \quad (a)$$
For the case of axially symmetric deformation, equation (a) is

\[
\frac{d^4 W}{dr^4} + \frac{2}{r} \frac{d^3 W}{dr^3} - \frac{1}{r^2} \frac{d^2 W}{dr^2} + \frac{1}{r^3} \frac{dW}{dr} = \frac{q_3}{D}.
\] (b)

Show that for the case when \( q_3 = \text{const.} \) and when the plate is simply supported

\[ W(r = R) = 0; \quad M_r(r = R) = -D \left( \frac{\partial^2 W}{\partial r^2} + \frac{\nu}{r} \frac{\partial W}{\partial r} \right)_{r=R} = 0, \] (c)

the solution of (b) is

\[ W = \frac{q_0 R^4}{64D} \left( \frac{5 + \nu}{1 + \nu} - \frac{3 + \nu}{1 + \nu} \frac{r^2}{R^2} + \frac{r^4}{R^4} \right). \] (d)
Chapter 9

Pressure Between Two Bodies in Contact

9.1 Introduction

In this chapter we discuss a special problem of elasticity theory, the so-called contact problem. Suppose that two elastic bodies are in contact at a single point. If the bodies are pressed with a force, say $F$, the contact, because of deformation, will not be in a single point but will be over a small area of the surface of each body. This surface is called the pressure surface. The curve that bounds the pressure surface is called the pressure contour. The basic problem of contact mechanics is to determine the pressure surface for a given force $F$ and the distribution of stresses within the pressure surface.

To solve the contact problem basic assumptions of the classical theory of contact due to Hertz (1892) are introduced. In presenting them we follow Johnson (1987)

1. Bodies in contact are linearly elastic homogeneous, with possibly different Lamé constants.

2. In the unloaded state the bodies are in contact at the single point $O$ that lies on a smooth part of the outer surface of both bodies. The point $O$ is a regular point of both surfaces and there is a well-defined tangent plane at $O$.

3. The bodies are compressed along the normal to the tangent plane at $O$.

We treat both static and dynamic problems. The dynamic problem refers to the impact of two elastic bodies. For impact according to both the classical Hertz theory and modified theory, also see Klichevski (1976).

9.2 Hertz’s Problem and Its Solution

We begin with geometrical preliminaries. Consider two bodies $B_1$ and $B_2$ in contact as shown in Fig. 1. Let $\Pi$ be the tangent plane to the outer surfaces of both bodies at the point of contact in the unloaded state. Let $\tilde{x}_1$, $\tilde{x}_3^{(1)}$ and $\tilde{x}_1$, $\tilde{x}_3^{(2)}$ be two Cartesian coordinate systems with the
origin at $O$ with the axes $\bar{x}_1$, $\bar{x}_1$ in tangent plane $\Pi$ and $\bar{x}_3^{(1)}$, $\bar{x}_3^{(2)}$ axes coincident with the inward normal to the surface of the bodies $B_1$ and $B_2$, respectively (see Fig. 1).

![Figure 1](image)

The equations of the outer surfaces of the bodies $B_1$ and $B_2$ are expanded in a Taylor series at the point of contact $O$. The result is

$$
\begin{align*}
  x_3^{(1)} &= \frac{1}{2} \left( \frac{\partial^2 f_1}{\partial x_1^2} \right)_{(0,0)} x_1^2 + \frac{1}{2} \left( \frac{\partial^2 f_1}{\partial x_2^2} \right)_{(0,0)} x_2^2 + \left( \frac{\partial^2 f_1}{\partial x_1 \partial x_2} \right)_{(0,0)} x_1 x_2; \\
  x_3^{(2)} &= \frac{1}{2} \left( \frac{\partial^2 f_2}{\partial x_1^2} \right)_{(0,0)} x_1^2 + \frac{1}{2} \left( \frac{\partial^2 f_2}{\partial x_2^2} \right)_{(0,0)} x_2^2 + \left( \frac{\partial^2 f_2}{\partial x_1 \partial x_2} \right)_{(0,0)} x_1 x_2.
\end{align*}
$$

(1)

Note that we can always choose axes $u_1$ and $v_1$ in $\Pi$ so that

$$
x_3^{(1)} = \frac{1}{2} [K_{11} u_1^2 + K_{12} v_1^2],
$$

(2)

where $K_{11}$ and $K_{12}$ are principal curvatures of the surface of $B_1$ given as $K_{11} = (\partial^2 f_1 / \partial u_1^2)_{(0,0)}$ and $K_{12} = (\partial^2 f_1 / \partial v_1^2)_{(0,0)}$. Similarly there are axes $u_2$ and $v_2$ in $\Pi$ such that

$$
x_3^{(2)} = \frac{1}{2} [K_{21} u_2^2 + K_{22} v_2^2],
$$

(3)

where $K_{21} = (\partial^2 f_2 / \partial u_2^2)_{(0,0)}$ and $K_{22} = (\partial^2 f_2 / \partial v_2^2)_{(0,0)}$. Let $\Psi$ be the angle between the $u_1$ and $u_2$ axes. We introduce a new coordinate system with
the axes u and v. Let \( \Psi_1 \) be the angle between \( u \) and \( u_1 \). Then \( \Psi_2 = \Psi_1 - \Psi \) is the angle between \( u \) and \( u_2 \). With this notation the coordinates of an arbitrary point on the surfaces in contact are connected as

\[
\begin{align*}
 u_1 &= u \cos \Psi_1 - v \sin \Psi_1; & v_1 &= u \sin \Psi_1 + v \cos \Psi_1; \\
 u_2 &= u \cos \Psi_2 - v \sin \Psi_2; & v_2 &= u \sin \Psi_2 + v \cos \Psi_2. 
\end{align*}
\]

By using (4) equations (2) and (3) become

\[
\begin{align*}
 x_3^{(1)} &= \frac{1}{2} [u^2(K_{11} \cos^2 \Psi_1 + K_{12} \sin^2 \Psi_1) + v^2(K_{11} \sin^2 \Psi_1 + K_{12} \cos^2 \Psi_1) \\
 &\quad - uv(K_{11} - K_{12}) \sin 2\Psi_1 ]; \\
 x_3^{(2)} &= \frac{1}{2} [u^2(K_{21} \cos^2 \Psi_2 + K_{22} \sin^2 \Psi_2) + v^2(K_{21} \sin^2 \Psi_2 + K_{22} \cos^2 \Psi_2) \\
 &\quad - uv(K_{21} - K_{22}) \sin 2\Psi_2 ].
\end{align*}
\]

Note that the distance between two arbitrary points \( M_1 \) and \( M_2 \) (see Fig. 1) in the unloaded state is \( x_3^{(1)} + x_3^{(2)} \). The orientation of the axes \( u \) and \( v \) (i.e., the angles \( \Psi_1 \) and \( \Psi_2 = \Psi_1 - \Psi \)) could be chosen so that

\[
(K_{11} - K_{12}) \sin 2\Psi_1 + (K_{21} - K_{22}) \sin 2\Psi_2 = 0. \quad (6)
\]

Let

\[
A = \frac{1}{2} [K_{11} \cos^2 \Psi_1 + K_{12} \sin^2 \Psi_1 + K_{21} \cos^2 \Psi_2 + K_{22} \sin^2 \Psi_2];
\]

\[
B = \frac{1}{2} [K_{11} \sin^2 \Psi_1 + K_{12} \cos^2 \Psi_1 + K_{21} \sin^2 \Psi_2 + K_{22} \cos^2 \Psi_2], \quad (7)
\]

and suppose that (6) is satisfied. Then the distance between \( M_1 \) and \( M_2 \) becomes

\[
x_3^{(1)} + x_3^{(2)} = Au^2 + Bv^2. \quad (8)
\]

We prove next that \( A > 0 \) and \( B > 0 \). To do this note that from (7) it follows that

\[
A + B = \frac{1}{2} [K_{11} + K_{12} + K_{21} + K_{22}];
\]

\[
B - A = \frac{1}{2} [(K_{11} - K_{12})^2 + (K_{22} - K_{21})^2 \\
+ 2(K_{11} - K_{12})(K_{21} - K_{22}) \cos 2\Psi]^{1/2}. \quad (9)
\]

By introducing the angle \( \tau \) by

\[
\cos \tau = \frac{B - A}{B + A}, \quad (10)
\]
we obtain
\[ A = B \tan^2(\tau/2); \quad A + B = \frac{A}{\sin^2(\tau/2)}. \]  
(11)

From (11) we have
\[ A = \frac{1}{2} [K_{11} + K_{12} + K_{21} + K_{22}] \sin^2(\frac{\tau}{2}); \]
\[ B = \frac{1}{2} [K_{11} + K_{12} + K_{21} + K_{22}] \cos^2(\frac{\tau}{2}). \]  
(12)

Thus \( A \) and \( B \) are of the same sign. Since the distance given by (8) is positive, it follows that \( A \) and \( B \) are positive. From (8) we conclude that the curves of equal distance between the points of the contacting surfaces laying on the same normal to the plane \( \Pi \) are concentric ellipses.

Suppose now that the bodies are pressed to each other by the force \( F \) directed along the normal to \( \Pi \) that passes through the point \( O \). Because of deformation the bodies will be in contact over a small surface called the pressure surface. It may be assumed, with sufficient accuracy, that the bodies come in contact at points laying before deformation on the same normal to \( \Pi \). From (8) we conclude that the pressure surface has an elliptical shape.

The next assumption is about deformation of the bodies at points far from the pressure surface. Namely, we neglect deformation at such points. Then, as a result of compression, any two points laying on the \( \vec{x}_3^{(1)}, \vec{x}_3^{(2)} \) axes sufficiently far from the point \( O \) will come closer to each other by an amount \( \alpha \) that is equal to the sum of the displacements of the point \( O \) considered as a point belonging to the bodies \( B_1 \) and \( B_2 \),
\[ \alpha = u_{30}^{(1)} + u_{30}^{(2)}, \]  
(13)

where \( u_{30}^{(1)} = u_3^{(1)} \) \((u = 0, v = 0)\), \( u_{30}^{(2)} = u_3^{(2)} \) \((u = 0, v = 0)\). Let \( u_3^{(1)} \) and \( u_3^{(2)} \) be the components of the displacement vector along the \( \vec{x}_3^{(1)}, \vec{x}_3^{(2)} \) axes, respectively, of two arbitrary points (say \( M_1 \) and \( M_2 \)) laying on the same normal to \( \Pi \). The distance between these two points decreases for an amount \( \alpha - (\vec{u}_3^{(1)} + \vec{u}_3^{(2)}) \). Therefore the pressure surface (initial distance is equal to the decrease in distance) is characterized by
\[ \vec{x}_3^{(1)} + \vec{x}_3^{(2)} = \alpha - (\vec{u}_3^{(1)} + \vec{u}_3^{(2)}). \]  
(14)

By using (8) in (14) we obtain
\[ u_3^{(1)} + u_3^{(2)} = \alpha - Au^2 + Bu^2. \]  
(15)

The points outside the pressure surface are characterized by
\[ \vec{x}_3^{(1)} + \vec{x}_3^{(2)} > \alpha - (\vec{u}_3^{(1)} + \vec{u}_3^{(2)}). \]  
(16)
We assume that the pressure surface is small compared to the dimensions of the body in contact so that the bodies in contact may be approximated by an elastic half space. Also we assume that there is no friction between the bodies in contact (the tangential component of the stress vector on the pressure surface is equal to zero). Then we may use (5.10-21) to determine the components of the displacement vector

\[
\begin{align*}
\mathbf{u}_3^{(1)} &= \frac{(1 - \nu_1)}{2\pi\mu_1} \int \int_{\Omega} \frac{q(\xi, \eta)d\Omega}{[(u - \xi)^2 + (v - \eta)^2]^{1/2}}; \\
\mathbf{u}_3^{(2)} &= \frac{(1 - \nu_2)}{2\pi\mu_2} \int \int_{\Omega} \frac{q(\xi, \eta)d\Omega}{[(u - \xi)^2 + (v - \eta)^2]^{1/2}}.
\end{align*}
\]

By using (17) in (14) we obtain

\[
\left[\frac{1 - \nu^2_1}{\pi E_1} + \frac{1 - \nu^2_2}{\pi E_2}\right] \int \int_{\Omega} \frac{q(\xi, \eta)d\Omega}{[(u - \xi)^2 + (v - \eta)^2]^{1/2}} = \alpha - Au^2 - Bv^2. \tag{18}
\]

In (17) and (18) \(\mu_1, \mu_2, \nu_1, \nu_2, E_1\) and \(E_2\) denote the shear modulus, Poisson ratios, and modulus of elasticity of the bodies \(B_1\) and \(B_2\), respectively. The solution of the Hertz problem is reduced to: find the pressure distribution \(q(\xi, \eta)\), the constant \(\alpha\) (the approach of the bodies), and size and shape of the pressure surface \(\Omega\) that satisfy (18). In (18) the constants \(A\) and \(B\) are known from the geometry of the bodies in contact. The improper integral in (18) represents the potential of a simple layer distributed with the density \(q(\xi, \eta)\) over the pressure surface (see Tychonoff and Samarski (1959))

\[
u_0 = \int \int_{\Omega} \frac{q(\xi, \eta)d\Omega}{r}. \tag{19}
\]

From (18) we conclude that the integral (19) is a quadratic function of \(u\) and \(v\). It follows from this conclusion that the solution of (18) could be connected with the potential inside a homogeneous ellipsoid. It is known that this potential is a quadratic function of coordinates. By using this idea Hertz obtained the solution of (18) from the potential of a homogeneous ellipsoid whose thickness in one direction tends to zero. On the basis of this analogy we assume that \(\Omega\) is an ellipse with axes \(a\) and \(b\) and that

\[
q(\xi, \eta) = \frac{3}{2\pi ab} \sqrt{1 - \frac{\xi^2}{a^2} - \frac{\eta^2}{b^2}}. \tag{20}
\]

Equation (18) then becomes

\[
(\beta_1 + \beta_2) \frac{3}{4} F \int_0^\infty \left(1 - \frac{u^2}{a^2 + \lambda} - \frac{v^2}{b^2 + \lambda}\right) \frac{d\lambda}{\sqrt{(a^2 + \lambda)(b^2 + \lambda)\lambda}} = \alpha - Au^2 - Bv^2. \tag{21}
\]
where
\[
\beta_1 = \frac{1 - \nu_1^2}{\pi E_1} = \frac{\lambda_1 + 2\mu_1}{4\pi\mu_1(\lambda_1 + \mu_1)}; \quad \beta_2 = \frac{1 - \nu_2^2}{\pi E_2} = \frac{\lambda_2 + 2\mu_2}{4\pi\mu_2(\lambda_2 + \mu_2)},
\] (22)
with \(\lambda_1, \mu_1, \) and \(\lambda_2, \mu_2\) being the Lamé constants of the bodies \(B_1\) and \(B_2\), respectively. Since (21) holds for all \(u\) and \(v\) we can use it to determine \(\alpha\). Thus by setting \(u = v = 0\) in (21) we obtain
\[
\alpha = \frac{3}{4}(\beta_1 + \beta_2) \int_0^\infty \frac{d\lambda}{\sqrt{(a^2 + \lambda)(b^2 + \lambda)}}.
\] (23)

Also by equating coefficients of \(u\) and \(v\) in (21) it follows that
\[
A = \frac{3}{4}(\beta_1 + \beta_2) \int_0^\infty \frac{d\lambda}{(a^2 + \lambda)(b^2 + \lambda)};
\]
\[
B = \frac{3}{4}(\beta_1 + \beta_2) \int_0^\infty \frac{d\lambda}{(a^2 + \lambda)(b^2 + \lambda)}.
\] (24)

From (21) we can determine \(a\) and \(b\) when \(A\) and \(B\) are known. With \(a\) and \(b\) known from (23) the “approach” of the bodies in contact \(\alpha\) follows.

By integrating (23),(24) we obtain (see Demidov (1979) and Johnson (1987) for details)
\[
\alpha = \frac{3}{2}(\beta_1 + \beta_2) \frac{F}{a} K(k); \quad A = \frac{3}{2}(\beta_1 + \beta_2) \frac{F}{a^3} \frac{K(k) - E(k)}{k^2};
\]
\[
B = \frac{3}{2}(\beta_1 + \beta_2) \frac{F k_0 [E(k) - K(k) k_0^2]}{b^3}.
\] (25)

where
\[
k_0 = \frac{b}{a} < 1; \quad k = \sqrt{1 - k_0^2},
\] (26)
and \(K(k)\) and \(E(k)\) are complete elliptic integrals of the first and second kind, respectively (see Lebedev (1972))
\[
K(k) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}; \quad E(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \phi} d\phi.
\] (27)

From (25) after some elementary transformations it follows that
\[
a = m\sqrt{\frac{3}{2} \frac{\beta_0}{\kappa}} F; \quad b = n\sqrt{\frac{3}{2} \frac{\beta_0}{\kappa}} F; \quad \alpha = \frac{K(k)}{\pi m} \sqrt{\frac{9}{4} \frac{\beta_0^2}{\kappa} F^2},
\] (28)

where
\[
\beta_0 = \pi(\beta_1 + \beta_2); \quad \kappa = 2(A + B) = K_{11} + K_{12} + K_{21} + K_{22};
\]
\[
m = \sqrt{\frac{2E(k)}{\pi(1 - k^2)}}; \quad n = m\sqrt{1 - k^2}.
\] (29)
If the principal radii of curvature $R_{11}, R_{12}, R_{21}, R_{22}$ are given, then principal curvatures $K_{11}, K_{12}, K_{21}, K_{22}$ are determined as

\[ K_{ij} = \pm 1/R_{ij}. \]  

(30)

The + sign is chosen if the surface is above its tangent plane and – if the surface is below its tangent plane. The maximal value of the pressure is determined from (20) for $\xi = \eta = 0$. The result is

\[ q_0 = (q(\xi, \eta))_{\text{max}} = \frac{3}{2} \frac{F}{\pi ab} = \frac{1}{\pi mn} \sqrt{\frac{3}{2} \left( \frac{\kappa}{\beta_0} \right)^2} F. \]  

(31)

With (31) we can write (20) as

\[ q(\xi, \eta) = q_0 \frac{\zeta}{o}, \]  

(32)

where

\[ \frac{\zeta}{o} = \sqrt{1 - \frac{\xi^2}{a^2} - \frac{\eta^2}{b^2}}. \]  

(33)

### 9.3 Examples of Contact Stresses

In this section we present results for some concrete cases of contact between elastic bodies.

1. **Elastic spheres**

Suppose the bodies in contact are two spheres of radius $R_1$ and $R_2$. Then, from (28) we have

\[ A = \frac{1}{R_1}; \quad B = \frac{1}{R_2}. \]  

(1)

Because of symmetry we have $a = b$. From (9.2-26) it follows that

\[ k = 0, \]  

(2)

so that (9.2-27) leads to

\[ K(k = 0) = \frac{\pi}{2}; \quad E(k = 0) = \frac{\pi}{2}. \]  

(3)

Also

\[ \kappa = 2 \left[ \frac{1}{R_1} + \frac{1}{R_2} \right]. \]  

(4)
By substituting (1) through (4) into (9.2-28), (9.2-31) we obtain

\[ a = b = \sqrt[3]{\frac{3}{4} \beta_0 F} \left( \frac{1}{R_1} + \frac{1}{R_2} \right), \]  

(5)

and

\[ \alpha = \frac{1}{2} \sqrt[3]{\frac{9}{2} \beta_0^2 \left( \frac{1}{R_1} + \frac{1}{R_2} \right) F^2}; \quad q_0 = \frac{3}{2} \frac{F}{\pi a^2} = \frac{1}{\pi} \sqrt[3]{6 \frac{1}{\beta_0^2} \left( \frac{1}{R_1} + \frac{1}{R_2} \right)^2 F}. \]

(6)

In the special case of steel spheres made of the same material \( E = E_1 = E_2 \), \( v_1 = v_2 = 0.3 \), we obtain

\[ a = b = 1.109 \sqrt[3]{\frac{F R_1 R_2}{E R_1 + R_2}}; \]

\[ \alpha = 1.231 \sqrt[3]{\frac{F^2}{E^2} \left( \frac{R_1 + R_2}{R_1 R_2} \right)}; \]

\[ q_0 = \frac{3}{2} \frac{F}{\pi a^2} = 0.388 \sqrt[3]{\frac{F E^2}{R_1 R_2} \left( \frac{R_1 + R_2}{R_1 R_2} \right)^2}. \]

(7)

Suppose now that a sphere \( R_1 = R \) is in contact with the elastic half space. Then, \( R_2 = \infty \) so that

\[ A = \frac{1}{R}; \quad B = 0. \]

(8)

With (8) we obtain

\[ a = b = \sqrt[3]{\frac{3}{4} \beta_0 F R}; \quad \alpha = \frac{1}{2} \sqrt[3]{\frac{9}{2} \beta_0 F^2 R}; \quad q_0 = \frac{3}{2} \frac{F}{\pi a^2} = \frac{1}{\pi} \sqrt[3]{6 \frac{1}{\beta_0^2} \frac{F}{R^2}}. \]

(9)

If the half plane is rigid, (i.e., \( E_2 = \infty \)) we have (see (9.2-22) and (9.2-29)

\[ \beta_0 = \frac{1 - \nu^2}{E}, \]

(10)

where \( E \) is the modulus of elasticity and \( \nu \) is the Poisson ratio of the material of the sphere.

2. Two circular prismatic cylinders

Suppose two elastic cylinders, shown in Fig. 2 are in contact. The cylinders are loaded by a force \( F \). In this case

\[ K_{11} = \frac{1}{R_1}; \quad K_{12} = 0; \quad K_{21} = \frac{1}{R_2}; \quad K_{22} = 0. \]

(11)
We assume that the pressure surface is rectangular. Then in the formulae of Section 9.2 we use additional condition $a/b \to \infty$, $k \to 1$. With this assumption (9.2-32), (9.2-33) become

$$q(\xi, \eta) = q_0 \sqrt{1 - \frac{\eta^2}{b^2}}. \quad (12)$$

Let $p$ be the force per unit length of the pressure surface, measured along the axis of the cylinder, (i.e., $p = F/l$) (see Fig. 2). Then,

$$p = \int_{-b}^{b} q d\eta = q_0 \int_{-b}^{b} \sqrt{1 - \frac{\eta^2}{b^2}} \, d\eta = \frac{\pi}{2} q_0 b. \quad (13)$$

Figure 2

From (13) we obtain

$$q_0 = (q(\xi, \eta))_{\text{max}} = \frac{2p}{\pi b}. \quad (14)$$

The force $F$ can now be determined from (9.2-31) and (14) as

$$F = \frac{2}{3} \pi abq_0 = \frac{4}{3} ap. \quad (15)$$

It remains to determine $b$. We start from (15) and (9.2-28). Thus,

$$a = \sqrt[3]{\frac{2E(k)}{\pi(1-k^2)}} \sqrt[3]{\frac{3}{2} \frac{\beta_0}{\kappa} \frac{4}{3} ap}, \quad b = \sqrt{1-k^2} \sqrt[3]{\frac{2E(k)}{\pi(1-k^2)}} \sqrt[3]{\frac{3}{2} \frac{\beta_0}{\kappa} \frac{4}{3} ap}, \quad (16)$$
or
\[ a = \sqrt{\frac{4E(k)\beta_0}{\pi(1 - k^2)\kappa}}, \quad b = \sqrt{\frac{4E(k)\beta_0}{\pi\kappa}}p. \] (17)

By taking the limit when \( k \to 1 \) in (17) it follows that \( a \to \infty \) and (17) leads to
\[ b = \sqrt{\frac{4\beta_0}{\pi\kappa}}p = \sqrt{\frac{4\pi\beta_0p}{R_1R_2}} \frac{R_1R_2}{R_1 + R_2}. \] (18)

We conclude this section with a comment concerning the problem of determining stresses in the bodies in contact at points far from the pressure surface \( \Omega \). To determine those stresses we may use the results of Section 5.10. Specifically in (5.10-11) we can replace \( F \) by the elementary force \( dF = (qdn) \) and then integrate over the pressure surface \( \Omega \) to obtain the components of the stress tensor. Such results are presented, for example, in Demidov (1979).

### 9.4 Theory of Elastic Impact

We consider the problem of two elastic spheres moving towards each other by velocity \( v_0 \). The original theory of impact was formulated by Newton. He introduced the concepts of *perfectly elastic* and *imperfectly elastic* impact. Hertz’s theory of elastic impact, that we present, belongs to the class of perfectly elastic impacts since there is no dissipation of energy during the impact.

Hertz’s theory of impact is a local one since it takes into account the local deformation of the bodies only. It is based on Hertz’s results for static theory of contact stresses, presented in Section 9.2 with additional restrictions. The basic additional restriction concerns the relative velocity \( v_0 \) of the approaching bodies. We show, in the analysis that follows, that the contact time (i.e., time interval in which the contact force between the bodies is different from zero) depends on \( v_0 \). The velocity \( v_0 \) is assumed to be so small that the contact time is much larger than the period of free vibrations of the colliding bodies.

To obtain the differential equations describing the relative motion of the colliding bodies suppose that the bodies moving towards each other at the time instant \( t_0 = 0 \) come into contact. We assume further that the relative velocity of the bodies has only the component along the common normal at the point of impact. Let \( v_0 \) be this velocity. If we choose a coordinate system as we did in previous section, then
\[ m_1 \frac{d^2 x_{3c}^{(1)}}{dt^2} = F; \quad m_2 \frac{d^2 x_{3c}^{(2)}}{dt^2} = F, \] (1)

where \( m_1 \) and \( m_2 \) are masses of the colliding bodies, \( x_{3c}^{(1)} \) and \( x_{3c}^{(2)} \) are coordinates of the mass centers of the colliding bodies, and \( F \) is the component
of the resultant force on the direction of the common normal at the impact point (the direction of the axes $\mathbf{r}_3^{(1)}$ and $\mathbf{r}_3^{(2)}$). In writing (1) we made an additional assumption. Namely, (1) is valid if all points outside the small region have the same velocity. Thus, deformation is concentrated on the small area around the contact point (inertia of this part is neglected) and the rest of the body moves as a rigid body (inertia is not neglected but deformation is neglected).

Suppose that at the initial moment when the bodies come into contact, the distances between the mass centers of the bodies are $x_{3c0}^{(1)}$ and $x_{3c0}^{(2)}$. By using the assumption stated in Section 9.2, that is, by neglecting all deformations of the bodies but the local ones, we obtain

$$x_{3c}^{(1)} + x_{3c}^{(2)} = x_{3c0}^{(1)} + x_{3c0}^{(2)} - \alpha,$$  

(2)

where $\alpha$ is the approach of the bodies. From (2) and (1) we obtain

$$-m_1 \left( \frac{d^2 \alpha}{dt^2} + \frac{d^2 x_{3c}^{(2)}}{dt^2} \right) = F.$$  

(3)

Now (3) and (1)_2 lead to

$$\frac{m_1 m_2}{m_1 + m_2} \frac{d^2 \alpha}{dt^2} = -F.$$  

(4)

Equation (4) could be solved if we know the dependence of $F$ on $\alpha$. Following the analysis of Hertz (1892) we assume that during the impact the $F - \alpha$ relation is determined from the static analysis presented in Section 9.2. Thus, from (9.2-28) and (9.3-6) we conclude that

$$\alpha = kF^{2/3}, \quad k = \frac{1}{2} \sqrt{\frac{9}{2} \beta_0^2 \left( \frac{1}{R_1} + \frac{1}{R_2} \right)}.$$  

(5)

where $\beta_0$ is given by (9.2-29). With (5) and $k_1 = (m_1 + m_2)/(m_1 m_2)$, equation (4) becomes

$$\frac{d^2 \alpha}{dt^2} = -k_1 k_2 \alpha^{3/2},$$  

(6)

where $k_2 = (k)^{-3/2}$. From (6) and with the notation $\dot{\alpha} = d\alpha/dt$, we get

$$\frac{1}{2} (\dot{\alpha}^2 - v_0^2) = -\frac{2}{5} k_1 k_2 \alpha^{5/2}.$$  

(7)

From (7) it follows that there is no dissipation during the impact, since at the end of the contact time (i.e., $t = t^*$) we have $\alpha = 0$, and the intensity of the relative velocity vector is the same $\dot{\alpha}^2(t = t^*) = v_0^2$, or $\dot{\alpha}(t^*) = -v_0$. 


Also, from (7) the value of $\alpha$ when $\dot{\alpha} = 0$ and the corresponding compression force are

$$\alpha_{\text{max}} = \left(\frac{5}{k_1k_2}\right)^{2/5} \left(\frac{v_0}{2}\right)^{4/5}; \quad F_{\text{max}} = k_2 \left(\frac{5}{4} \frac{v_0^2}{k_1k_2}\right)^{3/5}. \quad (8)$$

The contact time (i.e., duration of the impact) is

$$T_{\text{max}} = 2 \int_0^{\alpha_{\text{max}}} \frac{d\alpha}{\sqrt{\frac{v_0^2}{4}k_1k_2\alpha^{5/2}}}. \quad (9)$$

By using (8) to obtain $v_0$ in terms of $\alpha_{\text{max}}$ and by setting $\alpha = \xi\alpha_{\text{max}}$ equation (9) could be written as

$$T_{\text{max}} = 2\frac{\alpha_{\text{max}}}{v_0} \int_0^1 \frac{d\xi}{\sqrt{1-\xi^{5/2}}} \quad (10)$$

Then by the substitution $u = \xi^{5/2}$ we transform the integral in (10) as

$$\int_0^1 \frac{d\xi}{\sqrt{1-\xi^{5/2}}} = \frac{2}{5} \int_0^1 u^{-3/5}du = \frac{2}{5} \frac{\Lambda \left(\frac{5}{3}\right)}{\Lambda \left(\frac{9}{10}\right)}, \quad (11)$$

where $\Lambda(x)$ is the gamma function. Since $\Lambda(1/2) = \sqrt{\pi}$ (see Abramowitz and Stegun (1965)) we obtain

$$T_{\text{max}} = \frac{4}{5} \sqrt{\frac{\pi}{\Lambda \left(\frac{9}{10}\right)}} \frac{\alpha_{\text{max}}}{v_0} = 2.9432 \frac{\alpha_{\text{max}}}{v_0}. \quad (12)$$

By substituting (8)$_1$ in (12) we finally obtain contact time as

$$T_{\text{max}} = \frac{4}{5} \sqrt{\frac{\pi}{\Lambda \left(\frac{9}{10}\right)}} \frac{1}{v_0} \left(\frac{5}{k_1k_2}\right)^{2/5} \left(\frac{v_0}{2}\right)^{4/5} = \psi v_0^{-1/5}. \quad (13)$$

The constant $\psi$ could be determined experimentally. Note that the contact time varies inversely as the fifth root of the relative velocity of approach of the bodies before the impact.

We now estimate the period of the free vibrations of the colliding bodies $T_{\text{max}}$ to obtain the bounds on $v_0$ that guarantee $\tau_{\text{max}} < < T_{\text{max}}$. This is done on the basis of results for free vibration of spheres (Love 1944 p. 285). By using for comparison the velocity of irrotational waves (see (5.14-6))

$$c_2 = \sqrt{\frac{\lambda + 2\mu}{\rho_0}}, \quad (14)$$
we can estimate the largest period of free vibration for an elastic sphere. For example, in the special case when $\lambda = \mu$ we have

$$\tau_{\text{max}} \approx \frac{2}{0.8159} \frac{R}{c_2},$$  \hspace{1cm} (15)$$

where $R$ is the radius of the sphere. Thus, the results obtained so far are valid when $\tau_{\text{max}}$ given by (12) is much larger than $\tau_{\text{max}}$ given by (15).

Equation (4) could be transformed to the universal form as suggested by Klichevski (1984). We integrate (4) twice to obtain

$$\alpha = v_0 t - \frac{1}{m} \int_0^t \left[ \int_0^u F(s) ds \right] du,$$  \hspace{1cm} (16)$$

where $m = 1/k_1$. Next we introduce the dimensionless time $\tau$ as

$$\tau = t \frac{v_0^{1/5}}{m^{2/5} k^{3/5}},$$  \hspace{1cm} (17)$$

where $k$ is given by (5) and the dimensionless force $f$,

$$f(\tau) = \left( \frac{mv_0^2}{k} \right)^{-3/5} F(t).$$  \hspace{1cm} (18)$$

Then (5) becomes

$$\alpha = k \left( \frac{mv_0^2}{k} \right)^{2/5} f^{2/3}(\tau).$$  \hspace{1cm} (19)$$

By using (17) and (19) in (16) we obtain

$$f^{2/3}(\tau) + \int_0^\tau \left[ \int_0^{\tau_1} f(\tau_2) d\tau_2 \right] d\tau_1 - \tau = 0.$$  \hspace{1cm} (20)$$

The integral equation (20) is universal in the sense that it does not contain any physical or geometrical parameter. With the substitution

$$g = f^{2/3},$$  \hspace{1cm} (21)$$

and differentiation, (20) becomes

$$\frac{d^2 g}{d\tau^2} + g^{3/2} = 0.$$  \hspace{1cm} (22)$$

The initial conditions corresponding to (22) are

$$g(0) = 0; \hspace{0.5cm} \frac{dg}{d\tau}(\tau = 0) = 1.$$  \hspace{1cm} (23)$$

By numerical integration of (22), (23) we obtain the nondimensional contact time as

$$g(\tau^*) = 0; \hspace{0.5cm} \tau^* = 3.21805.$$  \hspace{1cm} (24)
The solution of (20) was obtained by Klichevski (1984) in the form

\[ f(\tau) = \frac{-10\tau^{3/2}}{13.793 + 0.73\tau^{5/2}} \cdot \frac{-0.276 - 0.0157\tau^{5/2}}{0.2 + 0.013\tau^{5/2}}. \]  

(25)

Note that the nondimensional contact time obtained from (25) is \( \tau^* = 3.217 \).

From (16) and (17) we can calculate the relative velocity during the impact as

\[ v = \frac{d\alpha}{dt} = v_0 \left[ 1 - \int_0^\tau f(\tau_1)d\tau_1 \right]. \]  

(26)

If we define the coefficient of restitution as

\[ r = \frac{|v(\tau = 0)|}{|v(\tau = \tau^*)|}, \]  

(27)

then from (7) we have \( |v(\tau = \tau^*)| = v_0 \) so that

\[ r = 1. \]  

(28)

By using (28) in (26) we conclude that

\[ \int_0^{\tau^*} f(\tau)d\tau = 2. \]  

(29)

The condition (29) could be used to check numerical integration of (22), (23). In our case

\[ \int_0^{3.21805} f(\tau)d\tau = 2.0. \]  

(30)

We comment on the generalized theory that allows for larger initial velocities \( v_0 \) to be analyzed. By using Laplace's transform (see Section 4.6) applied to the Lamé equations (4.3-1) Klichevski (1984) showed how to take into account inertial forces. Villaggio (1996) used the solution for the heavy sphere kept in equilibrium by a vertical force (obtained by Bondareva (1970)) to estimate the influence on the impact of the deformation of the body outside a small region. By using Villaggio (1996) results our equation (22) becomes

\[ \frac{d^2g}{d\tau^2} + \Phi(g)g^{3/2} = 0, \]  

(31)

where, for the sphere of radius \( R \) colliding against the rigid plane, we have

\[ \Phi(g) = \left[ 1 - \frac{2}{3} \frac{(1 - 2\nu)^2}{(1 - \nu)} \frac{1}{\pi}(\beta g)^{1/2} \left\{ \ln 2 - \frac{2}{3} + \ln(\beta g)^{1/2} \right\} \right. \]

\[ + \left. \frac{1}{2} \frac{(1 - 2\nu)}{(1 - \nu^2)} \frac{1}{\pi}(\beta g)^{1/2} \right], \]  

(32)
and

\[ k = \left( \frac{3}{4E^*R^{1/2}} \right)^{2/3}; \quad E^* = \frac{1}{\beta_0} = \frac{E}{1 - \nu^2}; \quad \beta = \frac{k^{3/5}(m\nu_0^2)^{2/5}}{R}. \] (33)

The initial conditions corresponding to (32) are given by (23). By solving (32), (23) numerically for a few values of \( \beta \) it was shown in Atanackovic et al. (1998) that the maximal deformation of the sphere \( g_{\text{max}} \) increases with increasing \( \beta \). Note also that for the case when the sphere is made of incompressible material (i.e., \( \nu = 0.5 \)) equation (32) reduces to (22).

The contact of elastic bodies with adhesive forces was treated by Johnson (1987). Using the Johnson model, Atanackovic and Spasic (2000) treated the problem of elastic impact with adhesive forces. In this case the relevant system of equations, instead of (22), (23), becomes (in dimensionless form)

\[ \frac{d^2 \delta}{d\tau^2} = -P; \quad P = a^3 - \beta a^{3/2}; \quad \delta = a^2 - \frac{2}{3\beta a^{1/2}}, \] (34)

subject to

\[ \delta(0) = 0; \quad \frac{d\delta}{d\tau}(\tau = 0) = 1. \] (35)

In (34) \( \delta \) denotes the relative displacement of the centers of masses of the two bodies in impact, estimated from their relative position at time \( \tau = 0 \), \( P \) is the force between the bodies, \( a \) is the radius of the contact circle, and \( \beta \) is the coefficient describing the ratio of adhesive and elastic forces. Note that for \( \beta = 0 \) (34) reduces to (22). Also, note that in (34) the parameter \( a \) could not be eliminated so one is not able to obtain a unique \( a(\delta) \) and \( P(\delta) \). This leads to different compression and restitution paths, denoted by 0 – 1 – 2 – 3 and 3 – 4 – 5 – 6 respectively (see Fig. 3).

![Figure 3](image-url)

The system (34), (35) can model both rebound when the bodies separate after the impact (at point 5) and capture when relative motion of the bodies stops (at point 4). In such a case the bodies will remain in contact and perform an oscillatory motion after impact. For detailed numerical analysis of (34), (35) see Atanackovic and Spasic (2000).
Problems

1. On the elastic half space $x_3 \geq 0$ a rigid disc of radius $a$ is positioned. A concentrated force $F$ is applied on the disc (see figure). Assume that the contact between the disc and space is smooth so that there are no shear stresses at the pressure surface.

Determine the resulting stresses in the half space.

i) By using (5.10-20) show that the $x_3$ component of the displacement vector of the points under the discs (i.e., $x_3 = 0$) are

$$u_3(x_1, x_2, x_3 = 0) = \frac{1}{4\pi\mu} \int \int_\Omega \left[ \frac{2(1 - \nu)}{\sqrt{\xi_1^2 + \eta_2^2}} \right] q(\xi, \eta) d\Omega, \quad (a)$$

where $\Omega = \{ (\xi, \eta) : \xi^2 + \eta^2 = a^2 \}$. In (a) the function $q(\xi, \eta)$ satisfies the condition

$$q(\xi, \eta) = 0, \quad \text{for} \quad \sqrt{\xi^2 + \eta^2} > a.$$

ii) Observe that $u_3(x_1, x_2, x_3 = 0) = \text{const.}$ because of the rigidity of the disc. Also since the disc is loaded by force $F$, show that

$$u_3(x_1, x_2, x_3 = 0) = \frac{(1 - \nu^2) F}{\pi E} \int \int_\Omega \frac{q(\xi, \eta) d\Omega}{\sqrt{\xi_1^2 + \eta_2^2}} = \text{const.}$$

$$F = \int \int_\Omega q(\xi, \eta) d\Omega. \quad (b)$$

iii) Next show that the function

$$q(\xi, \eta) = \frac{q_0}{2\sqrt{1 - (r/a)^2}}; \quad r = \sqrt{\xi^2 + \eta^2}; \quad q_0 = \frac{F}{\pi a^2}, \quad (c)$$
solves (b). Note the stress singularity at the end of the disc. Also there is a slip between the disc and the elastic half space. The case of no slip boundary condition (adhesive boundary conditions) between disc and half plane is treated by Mosakovskii (1954).

2. For the case described in Problem 1 when the disc is loaded by eccentrically positioned force \( \mathbf{F} = Fe_3 \) whose point of application is \( (x_1 = e, x_2 = 0, x_3 = 0) \) with \( e < a \) the solution could be obtained by the same procedure as in Problem 1. Thus, show that

\[
q(r, \vartheta) = \frac{1 + 3\frac{er}{a^2} \cos \vartheta}{2\pi a \sqrt{a^2 - r^2}} F; \quad r = \sqrt{x_1^2 + x_2^2}; \quad \tan \vartheta = \frac{x_2}{x_1};
\]

\[
u_3(x_1, x_2, x_3 = 0) = u_3(r, \vartheta)
\]

\[
= \frac{(1 - \nu^2) F}{\pi E} \int \int q(\xi, \eta) d\Omega = A + Bx_1, \quad (a)
\]

where \( A \) and \( B \) are constants. By using (a)\(_1\) in (a)\(_4\) it follows that

\[
u_3(r, \vartheta) = \frac{(1 - \nu^2) F}{\pi E} \left[ \frac{3 er}{2 a^2} \cos \vartheta + 1 \right]. \quad (b)
\]

(see Rekach (1977 p. 172)).

The case of a rigid disc of radius \( A \) loaded by two forces \( \mathbf{V} \) and \( \mathbf{H} \) having the action lines perpendicular and parallel to the middle plane of disc and two couples \( \mathbf{M}_2 \) and \( \mathbf{M}_3 \) with the action lines directed along the \( \bar{x}_2 \) and \( \bar{x}_3 \) axes, respectively (see figure below) was treated by Alexandrov and Soloveev (1978). By assuming that there is no slip between the disc and the elastic half space (the adhesive boundary conditions) the displacement of the center of the disc \( (e \text{ and } d) \) and the rotation angles about \( \bar{x}_2 \) axis \( (\alpha) \) and \( \bar{x}_3 \) axis \( (\gamma) \) is given as

\[
d = \frac{\kappa - 1}{16\pi \mu \beta a} V; \quad \gamma = \frac{3}{16\mu a^3} M_3;
\]

\[
\alpha = \frac{3(\kappa - 1)(M_2 + \beta aH)}{32\pi \mu \beta a^3 (\beta^2 + 1)}; \quad e = \alpha \beta a + \frac{4\pi \beta + \kappa - 1}{32\pi \mu \beta a} H, \quad (c)
\]

where \( \kappa = 4 - 4\nu; \quad \beta = (\ln \kappa)/(2\pi) \) with \( \nu \) being Poisson’s ratio.
3. Derive the estimate (9.4-15) for a sphere of radius $R$. Thus, show that

i) Lamé equation (4.3-11) in the spherical coordinate system for radial vibrations reduces to a single equation

$$ (\lambda + 2\mu \frac{\partial \vartheta}{\partial \rho} - \rho_0 \frac{\partial^2 u_\rho}{\partial t^2} = 0, $$

where $\vartheta = (\partial u_\rho/\partial \rho) + 2u_\rho/\rho$.

ii) Assume the solution of (a) in the form

$$ u_\rho = CW(\rho) \cos(\omega t + \alpha), $$

where $C$, $\omega$ (the angular frequency), and $\alpha$ are constants. By using (b) in (a) show that

$$ \frac{d^2 W}{d \rho^2} + \frac{2}{\rho} \frac{dW}{d \rho} + \left( h^2 - \frac{2}{\rho^2} \right) W = 0, $$

where $h^2 = \rho_0 \omega^2 / (\lambda + 2\mu)$.

iii) Show that the boundary conditions corresponding to (c) follow from

$$ \sigma_{\rho r}(\rho = R) = 0, \quad u_\rho(\rho = 0) < \infty $$

(d)
and are

$$(\lambda + 2\mu)\frac{dW(\rho = R)}{d\rho} + 2\lambda\frac{W(\rho = R)}{R} = 0; \quad W(\rho = 0) < \infty. \quad (e)$$

The solution of (c) satisfying (e)$_2$ is

$$W(\rho) = C\frac{(h\rho)\cos h\rho - \sin h\rho}{(h\rho)^2}, \quad (f)$$

where $C$ is an arbitrary constant.

iv) Show that (e)$_1$ leads to

$$\frac{\tan hR}{(hR)} = \frac{1}{1 - \frac{\lambda + 2\mu}{\mu}(hR)^2}. \quad (g)$$

In Love (1944, p. 285) the first six roots of (g) are presented in the special case when $\lambda = \mu$. 
Chapter 10

Elastic Stability

10.1 Introduction

In this chapter we briefly discuss the problem of determining stability of a given equilibrium configuration of an elastic body. First we review the concept of stability. We stress that a mechanical system (such as an elastic body) cannot be investigated for its stability, but only certain states of it (motion or equilibrium). Therefore, when we say that a body is stable we mean that its specific and clearly indicated equilibrium configuration is stable. The word stable is derived from Latin stabilis, to stand. Basically it refers to the property of a state of a (mechanical) system to remain unchanged, or to change only a little, when the system is subjected to perturbations.

An elastic body that is in equilibrium may be viewed as a dynamic and as a static system. Every dynamic system has time as an independent variable and a set of dependent variables (position vectors, velocities, stresses, strains, etc.) whose values are given for each time instant. The type of perturbations to which a dynamic system may be exposed leads to the distinction of two basically different concepts of stability: stability and structural stability (coarseness, robustness).

In the first case of stability (also called Liapunov stability) we investigate how the specific state of the body behaves when the body is subjected to a change of initial conditions. Thus, the motion of the body and the influence of initial conditions on the motion are analyzed. In the second case of structural stability we examine how the body (basic system) behaves when changes in the system are introduced (e.g., an additional force, change in the shape of the body, change in properties, etc.). The system with changes (in the system itself) forms a close system. The basic system whose stability is investigated is structurally stable if close systems have, qualitatively, the same behavior as the basic system. Of course, we have to define what is meant by the close system and what is "qualitatively" the same (see Golubitsky and Schaeffer (1985)).

There are three more or less independent approaches to the stability problems of elastic bodies. In the next section we state definitions of stability and then apply them to concrete problems.
10.2 Definitions of stability

The stability problems for elastic bodies belong to the class of nonlinear problems of mechanics. Namely, there exists a big difference in the conclusions that could be reached from linear as compared to nonlinear stability analysis. The nonlinearity in the problem can be geometrical (nonlinear relation between deformation measures and components of the displacement vector), physical (due to the nonlinear constitutive equations) or both. The stability analysis of elastic bodies started with the famous Euler work in 1744. Euler used the static method to determine the stability boundary for axially loaded rods for several types of boundary conditions.

From Knops and Willkes (1973) and Atanackovic (1997) we state the following definitions of stability and instability according to the Euler energy and dynamic methods.

1. The stability and instability according to the Euler static method (also called the method of adjacent equilibrium configuration)

   **Definition 1.**

   An equilibrium configuration of an elastic body is **stable** if under given load and given boundary conditions there are no other neighboring (infinitesimally close) equilibrium configurations.

   **Definition 2.**

   An equilibrium configuration of an elastic body is unstable if under given load and given boundary conditions there exists at least one more neighboring equilibrium configuration.

   The smallest value of the parameters (loads) for which there exists an adjacent equilibrium configuration is called the critical value of the parameter.

   From the uniqueness theorem (see Section 4.4) it follows that (for the cases of boundary conditions studied there) there is a unique solution of the equilibrium equations. Thus, there is no possibility of existence of neighboring configurations so that all equilibrium configurations are stable. Thus the Euler method can give nontrivial results only for the case of nonlinear elastostatic problems.

   As a matter of fact, the rotation of the body element (see Section 2.4) must be taken into account in writing equilibrium equations if one is to analyze stability by the Euler method in the case of a materially linear (Hooke's law valid) elastic body. Since, in using the Euler method, we are investigating the conditions under which the equilibrium equations have more than one solution the methods of bifurcation theory are applicable to determine the critical values of parameters. Suppose that the equations describing equilibrium are written in the form

   \[ G(y, \lambda) = 0, \]  

   (1)
where $G$ is a nonlinear operator mapping Banach space $Y$ into another Banach space $Z$, $y$ is an $n$-dimensional vector describing the state of the body depending on the position (i.e., $y = \dot{y}(x)$), and $\lambda$ is an $m$-dimensional vector describing parameters (such as loads). Suppose further that the configuration whose stability we examine is $y_0 = 0$, and that

$$G(0, \lambda) = 0;$$

that is, $y_0 = 0$ is an equilibrium configuration for any value of $\lambda$. According to the Euler stability criteria the critical value of $\lambda = \lambda_{cr}$ is a value such that for $\lambda > \lambda_{cr}$ there exists at least one other than trivial ($y_0 = 0$) solution of (1). The point $(y_0 = 0 \times \lambda_{cr} \in Y \times R^m)$ is called a bifurcation point of $G(y, \lambda)$ (see Chow and Hale (1982)). Our goal is to determine bifurcation points of $G(y, \lambda)$. The smallest value of $\lambda$ for which there is bifurcation is $\lambda_{cr}$. There are several procedures to determine $\lambda_{cr}$. Suppose that we write (1) in the form

$$G(y, \lambda) = B(\lambda)y + N(y, \lambda),$$

where $B(\lambda)$ is a Frechet derivative of $G$ at $y_0 = 0$ (linearization of $G(y, \lambda)$) and $N(y, \lambda)$ is the nonlinear part of $G$ satisfying $N(y, \lambda) = o(\|y\|)$. Consider the problem

$$B(\lambda)y = 0.\quad (4)$$

Suppose that $B(\lambda)$ is a bounded linear operator. Suppose further that $\lambda_0$ is an eigenvalue of $B(\lambda)$; that is, there exists a nontrivial $w_0$ such that

$$B(\lambda_0)w_0 = 0.\quad (5)$$

The following important result is proved in Chow and Hale (1982): *a necessary condition that $(0, \lambda_0)$ is a bifurcation point of $G(y, \lambda)$ is that $\lambda_0$ is an eigenvalue of $B(\lambda)$.*

In engineering applications this condition is usually taken as sufficient so that the smallest $\lambda_0$ satisfying (5) is taken as $\lambda_{cr}$. A sufficient conditions that the point $(0, \lambda_0)$ is a bifurcation point of the operator $G(y, \lambda)$ may be formulated in different ways. One of the best known results is given as the following theorem of Krasnoselski (1956): *if $\lambda_0$ is an eigenvalue of (5) of odd algebraic multiplicity then the point $(0, \lambda_0)$ is a bifurcation point of $G(y, \lambda)$.*

To define algebraic multiplicity of an eigenvalue, we note that $w_0$ belongs to the null space of $B(\lambda_0)$ that we denote by $N[B(\lambda_0)]$. The dimension of $N[B(\lambda_0)]$ (denoted by $\dim N[B(\lambda_0)]$) is called the geometric multiplicity of $\lambda_0$. It often happens for linear operators in Banach spaces that there exists an integer for which

$$\dim N[B(\lambda_0)^r] < \infty.\quad (6)$$

Let $r$ be such that

$$\dim N[B(\lambda_0)^{r+1}] = \dim N[B(\lambda_0)^r],\quad (7)$$
for every $r_1 \geq r$. Then, $r$ is called the \textit{algebraic multiplicity} of $\lambda_0$. We note that the condition that the eigenvalue is of odd algebraic multiplicity could be replaced by other conditions (see Antman (1995)).

2. The \textit{stability according to the energy method}

\textit{Definition 3.}

An equilibrium configuration of an elastic body under the action of conservative loads is stable if in this configuration the total potential energy of inner forces (i.e., stresses) and outer forces (body and surface) assumes a weak relative minimum in the class of virtual displacements.

\textit{Definition 4.}

An equilibrium configuration of an elastic body under the action of conservative loads is unstable if in this configuration the total potential energy is not in the minimum in the class of virtual displacements.

In Definitions 3 and 4 the virtual displacements are infinitesimal displacements satisfying kinematical constraints.\(^1\) It is important to note that the energy method is a \textit{static} method since it compares values of total energy in two configurations that are connected by virtual displacement. It is motivated by the Lagrange-Dirichlet theorem for mechanical systems with a finite number of degrees of freedom.

We state a necessary condition for a functional $f$ to have a weak relative minimum. Let $f : Y \to \mathbb{R}$ be a functional mapping Banach space $Y$ with norm $\| \cdot \|$ into set of real numbers $\mathbb{R}$. Suppose that $f$ has the second Gâteaux derivative $D^2 f(y_0; \eta, \eta)$ at the point $y_0 \in Y$ in the direction $\eta$, where $\eta = y - y_0$, $y \in Y$. Then, a necessary condition that $f$ has a weak relative minimum at $y_0$ is

$$D^2 f(y_0; \eta, \eta) \geq 0, \quad (8)$$

for all $\eta \in B_\varepsilon(0)$ where $B_\varepsilon = \{ \eta : \eta \in Y, \| \eta \| \leq \varepsilon \}.\(^2\) The sufficient condition for the existence of a minimum is

$$D^2 f(y_0; \eta, \eta) \geq \gamma(\| \eta \|) \| \eta \|, \quad (9)$$

where $\gamma(\tau)$ is a nonnegative continuous function on $(0, \infty)$ such that

$$\lim_{\tau \to \infty} \gamma(\tau) = +\infty. \quad (10)$$

\(^1\) For the definition of virtual displacement see Section 7.5.

\(^2\) Weak minimum is defined in the calculus of variations. It means that the norm on $Y$ is such that the “perturbed” functions $y$ are “close” to $y_0$ and that the gradient of $y$ is “close” to the gradient of $y_0$. This is achieved by the choice of the norm on $Y$. For example if $B_\varepsilon = \{ \eta : \eta \in Y, | \eta_i | + | \partial \eta_i / \partial x_j | \leq \varepsilon \}$ then (8) is a necessary condition for a weak minimum.
The condition (10) is often hard to verify. For that reason in engineering applications the critical value of the parameter is determined as follows. It is assumed that the configuration is stable as long as (8) is satisfied. The smallest value of parameters for which (8) is violated is termed the critical value.

3. The stability according to the dynamic method

To define stability according to the dynamic method we need some preliminary concepts. We define the so-called two-metric stability of Liapunov. As is seen from Chapter 4 the boundary value problems of elasticity and thermoelasticity could be written as a system of partial differential equations of the type

\[ \frac{\partial \chi_i}{\partial t} = \bar{F}_i(\chi_k, \mathbf{x}, t), \quad i = 1, 2, ..., n. \]  

(11)

In (11), \( n \) functions \( \chi_i(\mathbf{x}, t) \) are defined for each \( \mathbf{x} \in \mathbf{B} \), where \( \mathbf{B} \) is a part of \( \mathbf{E}^3 \) occupied by the body, for all values of time \( t \in [t_0, \infty) \). For any fixed \( t = t_1 \), the functions \( \chi_i(\mathbf{x}, t) \) define a configuration of the body and satisfy certain smoothness and boundary conditions. We say that for fixed \( t \) the functions \( \chi_i(\mathbf{x}, t) \) belong to a normed linear space \( \mathbf{Y} \). In our applications, \( \mathbf{Y} \) is certain Banach or Sobolev space. The initial conditions are specified in the form

\[ q_i(\mathbf{x}) = \chi_i(\mathbf{x}, t_0), \quad i = 1, 2, ..., n, \]  

(12)

and are imposed on the system (11). The initial conditions are required to belong to the subset \( \hat{\mathbf{Y}} \) of the space \( \mathbf{Y} \); that is, \( \hat{\mathbf{Y}} \subset \mathbf{Y} \). For the system (11), (12) we assume that the solution exists for all values of \( t \in [t_0, \infty) \) and that \( \chi_i(\mathbf{x}, t) \) belong to \( \mathbf{Y} \) for all \( t \in [t_0, \infty) \). If \( \hat{\chi}_k = \chi_i(\mathbf{x}) \), \( k = 1, 2, ..., n \) exist, so that

\[ \bar{F}_i(\hat{\chi}_k(\mathbf{x}), \mathbf{x}, t) = 0, \quad i = 1, 2, ..., n, \]  

(13)

we call \( \hat{\chi}_k = \hat{\chi}_k(\mathbf{x}) \) the equilibrium solution of (11). Since we are analyzing the stability of equilibrium states, in our applications equilibrium solutions will always exist. Suppose that the initial conditions of the body are subjected to perturbation. Let

\[ u_i(S, t) = \chi_i(\mathbf{x}, t) - \chi_i(\mathbf{x}, t_0), \]  

(14)

where \( \chi_j(\mathbf{x}, t_0) = \hat{\chi}_j(\mathbf{x}) \) is an equilibrium solution whose stability we examine. Note that for the case when there is no perturbation in the initial conditions, we have \( \chi_i = \hat{\chi}_i \), so that

\[ u_i(\mathbf{x}, t) \equiv 0. \]  

(15)

Otherwise, each perturbation of initial conditions of \( \chi_i \) defines an initial condition for \( u_i \). By using (14) in (11), we obtain

\[ \frac{\partial u_i}{\partial t} = G_i(u_k, \mathbf{x}, t), \quad i = 1, 2, ..., n, \]  

(16)
where \( G_i = \tilde{F}_i(u_k + \dot{\chi}_k, x, t) \). Let \( u = (u_1, u_2, ..., u_n) \). We call \( u \) the perturbation of the equilibrium solution. Then, (16) can be written as

\[
\frac{\partial u}{\partial t} = G(u, x, t), \tag{17}
\]

with \( u \in Y \). Here, \( Y \) is a normed linear space. The initial condition for \( u \) (i.e., perturbation of the initial conditions of \( \chi_i \)) we denote as

\[
u(x, t_0) = u_0(x). \tag{18}\]

In (18) we have \( u \in Y_0 \subset Y \). To define the distance between an element \( u \) and the null element \( u = 0 \) of \( Y \), we introduce a real functional (metric) \( \rho : Y \times [t_0, \infty) \to \mathbb{R} \) with the properties

\[
\rho(u, t) > 0, \quad \text{for } u \neq 0, \quad \rho(0, t) = 0. \tag{19}\]

In the case when \( \rho \) does not explicitly depend on \( t \), we write \( \tilde{\rho}(u) \). A functional satisfying (19) is called a metric. Note that the norm in the normed linear spaces can be used as a functional \( \rho \). Suppose two metrics \( \rho_0 \) and \( \rho \) are given, with \( \rho_0 = \tilde{\rho}_0(u) \) not depending on \( t \). We use \( \rho_0(u, t) \) to measure the perturbation at the initial moment \( t_0 \) and \( \rho(u, t) \) to measure the perturbation at any time instant \( t \in [t_0, \infty) \). We limit the function describing initial perturbation (perturbation of the initial conditions) \( u(x, t_0) = u_0(x) \) by requiring that the following inequality holds \( \rho_0(u(x, t_0)) = \rho_0(u_0(x)) < \delta \). Here, \( \delta \) is an arbitrary positive number. Note that \( \rho_0 \) and \( \rho \) can be used to define a neighborhood of \( u_0 = 0 \in Y_0 \) and \( u = 0 \in Y \), respectively. With given \( d_0 \) and \( d_1 \), closed balls of radius \( d_0 \) and \( d_1 \) at zero in \( Y_0 \) and \( Y \) are

\[
B_0(0, d_0) = \{ p : p \in Y_0, \rho_0(p) \leq d_0 \}; \\
B(0, d) = \{ p : p \in Y, \tilde{\rho}(p, t) \leq d \text{ for all } t \geq t_0 \}. \tag{20}\]

The metrics \( \rho_0 \) and \( \rho \) are not chosen completely arbitrarily. Namely, we require that \( \tilde{\rho}(u, t) \) be continuous with respect to \( t \) and that \( \tilde{\rho}(u, t) \) be continuous with respect to \( \rho_0 = \tilde{\rho}_0(u) \) for \( t = t_0 \) at \( \rho_0 = 0 \). The second requirement is defined as: given any \( \epsilon > 0 \) there exists \( \delta(\epsilon, t_0) > 0 \), such that \( \tilde{\rho}(u(S, t_0), t_0) < \epsilon \) if \( u_0(S) \) satisfies \( \rho_0(u_0(S, t_0)) < \delta(\epsilon, t_0) \). For example, if we take \( u = u, x = S, t_0 = 0 \), and

\[
\tilde{\rho}(u(S, t), t) = \int_0^L u^2(S, t) dS; \\
\rho_0(u(S, 0)) = \int_0^L \left[ u^2(S, 0) + \left( \frac{\partial u(S, 0)}{\partial S} \right)^2 \right] dS, \tag{21}\]

then \( \rho(u, t) \) is continuous with respect to \( \rho_0(u) \) at \( t_0 = 0 \), with \( \delta(\epsilon, t_0) = \epsilon \) (note that \( \rho_0(u) \) is not continuous with respect to \( \tilde{\rho}(u, t) \) at \( \tilde{\rho}(u, t) = 0 \)).
The condition of the continuity of $\bar{p}(u, t)$ with respect to $\bar{p}(u)$ for $t = t_0$ at $\bar{p}_0 = 0$ is introduced to avoid instability at the initial moment. In what follows we assume that $t_0 \in [0, T]$. We state the definition of stability with respect to two metrics $\bar{p}_0$ and $\bar{p}$ as:

**Definition 5.**

The trivial solution $u(S, t) \equiv 0$ of the system (17), (18) is Liapunov stable with respect to two metrics $\bar{p}_0$ and $\bar{p}$ if for each $\epsilon > 0$ there exists $\delta(\epsilon, t_0)$ such that

$$\bar{p}_0(u(S, t_0)) = \bar{p}_0(u_0(S)) < \delta(\epsilon, t_0),$$

implies

$$\bar{p}(u(S, t), t) < \epsilon.$$ (23)

In general, the number $\delta(\epsilon, t_0)$ is dependent on the choice of initial time instant $t_0 \in [0, T]$. If $\delta(\epsilon, t_0)$ is independent of $t_0$, the system is said to be uniformly stable.

**Definition 6.**

The trivial solution $u(S, t) \equiv 0$ of the system (17), (18) is Liapunov unstable with respect to two metrics $\bar{p}_0(u)$ and $\bar{p}(u, t)$ (recall that $\bar{p}(u, t)$ is required to be continuous with respect to time $t$ and continuous with respect to $\bar{p}_0(u)$ for $t = t_0$ at $\bar{p}_0 = 0$) if there exists a number $\epsilon > 0$ and an initial instant $t_0$, so that for every number $\delta$ no matter how small there exists at least one $\tilde{u}_0$ with $\bar{p}_0(\tilde{u}_0) < \delta$ such that $\bar{p}(u, t_1) \geq \epsilon$ for some time instant $t_1 \in [t_0, \infty)$, where $u$ is the solution of (17),(18) with $\tilde{u}_0$ as the initial condition.

**Definition 7.**

The trivial solution $u(x, t) \equiv 0$ of the system (17), (18) is asymptotically stable if and only if it is Liapunov stable with respect to two metrics $\bar{p}_0$ and $\bar{p}(u, t)$ and if there exists a positive real number $\delta$ depending on $t_0$, so that $\bar{p}_0(u_0) < \delta$ implies $\lim_{t \to \infty} \bar{p}(u(t), t) \to 0$, where $u$ is the solution of (17),(18) with $u_0$ as the initial condition.

Note that $u(S, t) \equiv 0$ may be stable with respect to one choice of $\bar{p}_0(u)$ and $\bar{p}(u, t)$, and unstable with respect to the other choice. Also, we may use only one metric $\bar{p}(u, t)$ and define $\bar{p}_0 = \bar{p}(u, t)|_{t=0}$ as metric for the initial perturbation.

Having defined three concepts of stability analysis we may ask: are all definitions of stability compatible? Is, the critical value of the load parameter the same when determined on the basis of the Euler, energy, and dynamic methods? If the critical values are the same, is it then possible to keep one definition of stability and derive the other two theorems? In
such attempts the dynamic method has a good chances to be taken as the
definition of stability and the value of other two methods judged accord­
ing to restrictions that have to be introduced in order to derive those two
methods from the dynamic method. We discuss those questions now.

If we want to compare the Euler method with the other two methods
this comparison can be done only for the class of problems to which all
three methods are applicable. Generally speaking, the Euler method is
applicable to problems in which the load is conservative. It could be shown
(see Atanackovic (1997)) that for nonconservative loads (Beck's column)
the Euler method may give erroneous results. It is interesting that for
some nonconservative loads, like those in the case of the Pflüger rod, the
Euler method gives the correct answer for the stability boundary. The other
limitation of the Euler method is the existence of snap-through. It could
happen (and does happen for shallow shells) that a small disturbance brings
the system from one equilibrium configuration to the other equilibrium
configuration that is not close to the initial one. The Euler method cannot
detect such instability since it tests only for adjacent configurations. Taking
all this into account it follows that for conservative systems (the Euler
method is applicable) when the total potential energy does not have a
proper minimum in the equilibrium configuration then another equilibrium
configuration exists in the neighborhood of this equilibrium configuration;
that is, there exists a weak solution to the linearized equilibrium equations
(Knops and Wilkes 1973 p. 224). That is, instability according to the energy
method implies instability according to the Euler method.

Before we compare the energy and dynamic methods note that sometimes
the energy definition of stability is stated as: an equilibrium configuration is
stable if the work done by loads on a virtual displacement compatible with
boundary conditions of place does not exceed the work of internal forces
(the increase of potential energy of internal forces). Otherwise, the configu­
ruration is called unstable. With such a definition, stability according to the
energy method is equivalent to the requirement that the second variation
of the total potential energy is nonnegative for all allowable superimposed
infinitesimal deformations. In our definition of stability according to the
energy method we ask for the proper local minimum of the total potential
energy. In such a case the positive second variation of the total potential en­
ergy is the only necessary condition for stability. Since the energy definition
of stability that we use (proper local minimum) puts stronger restrictions
on the total potential energy, we can compare the energy and dynamic
methods. In general, equivalence is proved for special choices of metrics
for the initial and subsequent disturbances and with some additional re­
sictions on total potential energy. Namely, the equivalence follows if we
use rather strong sufficient conditions for the existence of the weak local
minimum.

To conclude we may say that the Euler, energy, and dynamic methods
sometimes can, sometimes can not, be compared. Because of that we take
definitions of stability in all three methods as independent. The Euler and energy methods are static methods. That is, to answer the question, if an equilibrium configuration is stable in the analysis according to those two methods we use only properties of material (constitutive equations) that are relevant for static behavior. Those properties are much better understood than dynamic ones. On the other hand, when we use the dynamic method for stability analysis, we must know complete dynamic, constitutive equations to examine the stability of an equilibrium configuration. Thus we are led to the conclusion that the Euler and energy methods are "natural" methods for the analysis of stability of equilibrium states (Wang and Truesdell 1973).

10.3 Basic theorems of the dynamic method

We now present the basic theorems on two-metric stability of Liapunov. A. M. Liapunov has proposed the classification of the methods of stability analysis. In his classification two methods are distinguished: the first and the second method. The first method, also called the indirect method, is based on the analysis of differential equations of motion and properties of their solutions. The second, or direct method, is based on the analysis of differential equations of motion without using the solution of the differential equations. In this part we present elements of the second or direct method of Liapunov. For detailed treatment see, for example, Knops and Wilkes (1973), Zubov (1984), Sirazetdinov (1987), and Antman (1995). The fact that in using Liapunov second method we do not have to know the solution of the equations (we must know that the solution exists in a certain function space) makes the method very attractive. However, in applying it to practical problems we are often faced with subtle mathematical problems. Also, there is no general method of constructing the Liapunov functional, a quantity that is of central importance to the theory. We begin with definitions of the Liapunov functional.

Let \( V(u, t) \) be a functional \( V : Y \times [t_0, \infty) \to R \), which for fixed time instant \( t \in [t_0, \infty) \), maps \( u(\cdot, t) \in Y \) into the set of real numbers \( R \). Suppose further that the functional \( V \) satisfies \( V(0, t) = 0 \). Let \( B(0, d_1) \) be a closed ball in \( Y \) of radius \( d_1 \). Then, we have the following.

**Definition 1.**

The functional \( V(u, t) \) is positive (negative) definite on the closed ball \( B(0, d_1) \) with respect to the metric \( \bar{\rho} \), if it satisfies \( V \geq 0 \) \( (V \leq 0) \) for all \( t \in [t_0, \infty) \) and all \( u(\cdot, t) \in B(0, d_1) \subset Y \), and for an arbitrary \( 0 < \varepsilon < d_1 \) there exists a positive constant \( \delta(\varepsilon) \) such that \( V \geq \delta(\varepsilon) \) \( (-\delta(\varepsilon) \geq V) \) for all \( u(\cdot, t) \in B(0, d_1) \subset Y \) such that \( \bar{\rho}(u, t) \geq \varepsilon \).
Remark. From the definition it follows that the property of a functional to be positive definite depends on \( Y \). In application, the boundary conditions and the regularity of the elements of \( Y \) will be crucial in establishing positive definiteness. As a special case we may choose \( \delta(\epsilon) = \epsilon \) and \( \bar{\rho}(u,t) = \epsilon \), so that the functional \( V \) is positive definite with respect to \( \bar{\rho}(u,t) \) if \( V(u,t) \geq \bar{\rho}(u,t) \).

Definition 2.

The functional \( V(u,t) \) is continuous with respect to the metric \( \bar{\rho} \) at the time instant \( t = t_0 \in [0,T] \) and \( \bar{\rho}_0 = 0 \) if for an arbitrarily small positive number \( \epsilon > 0 \), a positive number \( \delta(\epsilon) \) exists so that \( |V(u(x,t_0),t_0)| < \epsilon \) is satisfied whenever \( \bar{\rho}_0 < \delta(\epsilon) \).

Remark. If the functional \( V \) is continuous with respect to \( \bar{\rho}_0 \) at \( t = t_0 \) and \( \bar{\rho}_0 = 0 \), and if \( |V| > \epsilon \) for some \( u(x,t_0) = \bar{\rho} \), then \( \bar{\rho}_0(\bar{\rho}) \geq \delta(\epsilon) \). To prove this, suppose that \( \bar{\rho}_0(\bar{\rho}) < \delta(\epsilon) \). The continuity of \( V \) then implies that \( |V| < \epsilon \), which is a contradiction.

We now state the principal theorems of stability (Movchan-Liapunov criteria for stability):

Theorem 1.

For the trivial solution \( u(x,t) \equiv 0 \) of the system of equations (10.2-17),(10.2-18) to be Liapunov stable with respect to the metrics \( \bar{\rho}_0 \) and \( \bar{\rho} \), it is necessary and sufficient that in a sufficiently small neighborhood \( B(0,d_1) \) of the point \( u(x,t) \equiv 0 \in Y \), there exists a functional \( V(u(x,t),t) \) with the properties:

1. \( V(u(x,t),t) \) is positive definite with respect to \( \bar{\rho} \);
2. \( V(u(x,t),t) \) is continuous with respect to \( \bar{\rho}_0 \) at \( t = t_0 \) and \( \bar{\rho}_0 = 0 \);
3. \( V(u(x,t),t) \) is not increasing (as a function of \( t \)) along the solutions of (10.2-17),(10.2-18).

Proof. First we show the necessity. Suppose that the system (10.2-17), (10.2-18) is Liapunov stable. We consider a neighborhood of zero \( u(x,t) \equiv 0 \) consisting of those elements \( u(x,t) \in Y \) for which we have \( \bar{\rho}(u(x,t),t) < d_1 \) if \( \bar{\rho}_0(u(x,t_0)) < d_0 \) for some values \( d_1 \) and \( d_0 \). Then, consider the functional

\[
V(u,t) = \sup_{\tau \in [t,\infty)} \bar{\rho}(u(x,\tau),\tau), \quad t \geq t_0 .
\]

The functional (1) satisfies all conditions stated in Theorem 1. To see this, note that

\[
V(u,t) \geq \bar{\rho}(u(x,t),t),
\]

so that \( V \) is the positive definite (with \( \delta = \epsilon \) in Definition 1 of this section). Since

\[
V(u(x,t_1),t_1) \geq V(u(x,t_2),t_2) \quad \text{if} \ t_1 \leq t_2,
\]
it follows that $V$ is not increasing. Note that with (1), the inequality (2) holds even if the solution $u(x,t) \equiv 0$ is not stable. To show that $V$ is continuous with respect to the metric $\rho_0$ at $t = t_0$ and $\rho_0 = 0$, we choose $\epsilon < d_1$. Since $u(x,t) \equiv 0$ is stable, there exist $\delta(\epsilon)$ such that the inequality $\rho_0(u) < \delta(\epsilon)$ implies $\rho(u(S,t),t) < \epsilon$ for $t > t_0$. Therefore,

$$\sup_{\tau \in [t,\infty)} \rho(u(x,\tau),\tau) = V(u,t) < \epsilon < d_1;$$  \hspace{1cm} (4)

that is, $V$ is continuous with respect to the metric $\rho_0$ at $t = t_0$ and $\rho_0 = 0$.

The sufficiency is proved as follows. Suppose that we are given the functional $V$ with the properties stated in Theorem 1. For given $\epsilon > 0$, we have to find $\delta(\epsilon)$ such that $\rho_0(u_0) < \delta$ implies $\rho(u,t) < \epsilon$ for all $t > t_0$.

Since $V$ is the positive definite with specified $\epsilon > 0$, we can find such $\delta_1(\epsilon) > 0$ that $V(u(t),t) < \delta_1(\epsilon)$ if $\rho(u(t)) < \delta$. Let $V$ be the positive definite functional with respect to $\rho$. Thus, with given $\epsilon > 0$, we can find such a constant $\delta_2 = \delta_2(\epsilon)$ that $\rho(u,t) < \epsilon$ at $t = t_0$ if $\rho_0(u_0) < \delta$.

Now, we use the fact that $\rho$ is continuous with respect to $\rho_0$ at $t_0$. Thus, with given $\epsilon > 0$, we can find such a constant $\delta_2 = \delta_2(\epsilon)$ that $\rho(u,t) < \epsilon$ at $t = t_0$ if $\rho_0(u_0) < \delta$.

Then, with given $\epsilon > 0$, there exist $\epsilon_1(\epsilon)$ and $\delta(\epsilon)$ such that $\rho(u,t) < \epsilon$ and $V(u,t) \leq \epsilon_1$ at $t = t_0$ if $\rho(u_0) < \delta(\epsilon)$. We use the fact that $V$ is nonincreasing to show that $\rho(u,t) < \epsilon$ holds for all $t > t_0$. Suppose now that $\rho(u,t) < \epsilon$ does not hold. Then, there exists a time instant $t_1 \in [t_0,\infty)$ such that $\rho(u,t_1) = \epsilon$. Since $V$ is the positive definite with respect to $\rho$, it follows that $V(u(S,t_1),t_1) > \epsilon_1(\epsilon)$. This is impossible since $V$ is nonincreasing and $V(u(u_0),t_0) < \epsilon_1$.

For the asymptotic stability the following theorem holds:

**Theorem 2.**

For the solution of (10.2-17),(10.2-18) to be asymptotically stable with respect to metrics $\rho_0$ and $\rho$, it is necessary and sufficient that it is Liapunov stable (see Theorem 1) and that

$$\lim_{t \to \infty} V(u(x,t),t) \rightarrow 0.$$  \hspace{1cm} (5)

**Proof:** We first prove the necessity. Suppose that $u(x,t) \equiv 0$ is asymptotically stable. We show that $V(u,t)$ satisfies (5). Since the asymptotic stability implies $\lim_{t \to \infty} \rho(u,t) = 0$, then for an arbitrarily small $\epsilon > 0$ such $t_1 < \infty$ exists that, for $t > t_1$ we have $\rho < \epsilon$. Therefore, with $V$ given by (1), we have $V = \sup_{\tau \in [t,\infty)} \rho(u(x,\tau),\tau) < \epsilon$ for $t > t_1$. Thus, $\lim_{t \to \infty} V = 0$.

Suppose now that the functional $V$ satisfying (5) exists and that $u(x,t) \equiv 0$ is Liapunov stable. Since $V$ is the positive definite functional with respect
to metric $\tilde{\rho}$, for given $\gamma > 0$, there exists such $\delta_1(\gamma) > 0$ that $V \geq \delta_1$ if $\rho \geq \gamma$. However since $\lim_{t \to \infty} V = 0$, there exists $\check{t} \in [t_0, \infty)$ such that $V(u(t), t) < \delta_1(\gamma)$ for $t > \check{t}$. We also claim that $\tilde{\rho}(u(t), t) < \gamma$ for $t > \check{t}$. To see this, suppose the contrary (i.e., such a sequence, say $\check{t} < t_1 < t_2 < \ldots < t_k < \infty$) exists that $\tilde{\rho}(u(x, t_i), t_i) > \gamma$, $i = 1, 2, \ldots, k$. Then since $V$ is positive definite, we have $V(u(x, t_i), t_i) > \delta(\gamma)$, for $i = 1, 2, \ldots, k$, which is a contradiction. Therefore, $\tilde{\rho}(u, t) < \gamma$ for $t > \check{t}$. Since $\gamma$ is arbitrary it follows that $\tilde{\rho}(u, t) \to 0$ as $t \to \infty$.

There is no general way that functional $V$ can be constructed. It is often the energy functional of the system, or some quantity related to energy (see Knops and Wilkes (1966), Zubov (1984), and Sirazetdinov (1987)). We finally note that two metrics $\rho_1$ and $\rho_2$ are equivalent (as a matter of fact, semi-equivalent) if such positive constants $c_1$ and $c_2$ exist that

$$c_1 \rho_1(u(x, t), t) \leq \rho_2(u(x, t), t) \leq c_2 \rho_1(u(x, t), t). \quad (6)$$

It can be shown that the stability (instability) with respect to the metrics $\rho_0$ and $\rho$ implies the stability (insatiability) with respect to $\rho_{10}$ and $\rho_1$ if $\rho_{10}$ is equivalent to $\rho_0$ and if $\rho_1$ is equivalent to $\rho$.

### 10.4 Examples

We present several examples of stability analysis.

1. **Buckling of a circular plate**

Consider a circular plate as shown in Fig. 1. Suppose that the plate has radius $R$ and that it is loaded so that the stress vector has a radial component of intensity $-T_0$. The plate is built in at its ends. By using von Kármán’s plate theory we examine the stability of the plate according to the Euler stability method. We assume that the deformation is axially symmetric and that there are no body forces acting; that is, $q_3 = 0$.

The differential equations of the problem are (see Problem 2 of Section 8).

$$D \frac{1}{r \, dr} \left\{ r \, \frac{d}{dr} \left[ \frac{1}{r \, dr} \left( \frac{dW}{dr} \right) \right] \right\} = \frac{1}{r \, dr} \left( r T_r \frac{dW}{dr} \right);$$

$$\frac{1}{Eh} \frac{r \, dr \, \left[ 1 \, \frac{d}{dr} (r^2 T_r) \right]} = - \frac{1}{2} \left( \frac{dW}{dr} \right)^2. \quad (1)$$

The boundary conditions corresponding to the plate shown in Fig. 1 are

$$\frac{dT_r(0)}{dr} = 0; \quad \frac{dW(0)}{dr} = 0;$$

$$T_r(r = R) = -T_0; \quad \frac{dW}{dr}(r = R) = 0. \quad (2)$$
The first two conditions express the symmetry of the problem. Note that

$$T_r = \frac{1}{r} \frac{d\Phi}{dr},$$

so that (1) becomes (see Wolkowisky (1967))

$$D \left\{ r \frac{d}{dr} \left[ \frac{1}{r} \frac{d}{dr} \left( r \frac{dW}{dr} \right) \right] \right\} = \frac{d\Phi}{dr} \frac{dW}{dr};$$

$$\frac{1}{Eh} r \frac{d}{dr} \left[ \frac{1}{r} \frac{d}{dr} (r^2 T_r) \right] = -\frac{1}{2} \left( \frac{dW}{dr} \right)^2. \quad (4)$$

We introduce the dimensionless quantities

$$\xi = \frac{r}{R}; \quad \lambda^2 = \frac{R^2 T_0}{D} = \frac{12(1 - \nu^2) R^2 T_0}{h^3 E};$$

$$q(\xi) = \left( \frac{hE}{T_0} \right)^{1/2} \frac{1}{R\xi} \frac{dW(\xi R)}{d\xi}; \quad p(\xi) = 1 + \frac{1}{R^2 T_0} \frac{1}{\xi} \frac{d\Phi(\xi R)}{d\xi}. \quad (5)$$

With (5) the system (4) becomes

$$\frac{1}{\xi^3} \frac{d}{d\xi} \left( \xi^3 \frac{dq}{d\xi} \right) + \lambda^2 (1 - p) q = 0; \quad \frac{1}{\xi^3} \frac{d}{d\xi} \left( \xi^3 \frac{dp}{d\xi} \right) = -\frac{1}{2} q^2, \quad (6)$$

subject to

$$\frac{dq(0)}{d\xi} = 0; \quad \frac{dp(0)}{d\xi} = 0; \quad q(1) = 0; \quad p(1) = 0. \quad (7)$$

Note that

$$q^0 \equiv 0; \quad p^0 \equiv 0,$$

is a solution (trivial) of (6),(7) for all values of $\lambda$. We want to determine the values of $\lambda$ for which there are nontrivial solutions that correspond to the buckled states of the plate. The linearization of (6) about (8) is

$$\frac{1}{\xi^3} \frac{d}{d\xi} \left( \xi^3 \frac{dq}{d\xi} \right) + \lambda^2 q = 0; \quad p = 0. \quad (9)$$
The nontrivial solutions of the linearized problem are
\[ \tilde{q}_n = \frac{C}{\xi} J_1(\lambda_n \xi), \tag{10} \]
where \( J_1 \) is the Bessel function of the first kind of the first order, \( C \) is an arbitrary constant, and \( \lambda_n \) are solutions of
\[ J_1(\lambda_n) = 0. \tag{11} \]
The smallest root of (11) is \( \lambda_1 = 3.831706 \). The corresponding load then becomes
\[ T_{0cr} = 14.68197 \frac{D}{R^2}. \tag{12} \]
Thus \((q = 0, p = 0, \lambda = 3.831706)\) defines a bifurcation point of (9). Since \( \lambda_1 \) is simple (geometric and algebraic multiplicities are equal to one) eigenvalue it follows that the system (6) also has \((q = 0, p = 0 \lambda_1 = 3.831706)\) as a bifurcation point; that is, there is loss of stability according to Euler criteria for \( \lambda > \lambda_1 \).

2. Stability of an elastic body with prescribed displacement vector and with no body forces

Consider a linear homogeneous elastic body in an equilibrium configuration. We take the relevant system of equations in the form (3.2-1), (3.3-47)
\[ \sigma_{ij} = C_{ijkl} E_{kl} = [\lambda \delta_{ij} \delta_{kl} + \mu(\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})] E_{kl} = \lambda \delta_{ij} + 2\mu E_{ij}. \tag{13} \]
We assume that the following boundary conditions are imposed
\[ u_i = 0 \quad \text{on } S, \tag{14} \]
that is, we are treating the second fundamental boundary value problem. We write (13) as (see 4.1-6)
\[ (\lambda + \mu) \frac{\partial \vartheta}{\partial x_i} + \mu \nabla^2 u_i = \rho_0 \frac{\partial^2 u_i}{\partial t^2}, \tag{15} \]
and examine stability of the trivial state
\[ u_i \equiv 0. \tag{16} \]
Let the metrics \( \rho_0(u_0) \) and \( \rho(u, t) \) be given as
\[ \rho_0 = \sup_{x \in V} |u_i(x, t = 0)| + \sup_{x \in V} \left| \frac{\partial u_i(x, t = 0)}{\partial x_j} \right| + \sup_{x \in V} \left| \frac{\partial u_i(x, t = 0)}{\partial t} \right|. \]
\[ \bar{\rho} = \int \int_V \left[ u_1^2 + u_2^2 + u_3^2 \right] dV. \] (17)

As the Liapunov functional we take the total mechanical energy of the body; that is, kinetic energy plus potential energy (see 7.8-5))

\[ V = \frac{1}{2} \int \int_V \int \left[ \rho_0 \frac{\partial u_i}{\partial t} \frac{\partial u_i}{\partial t} + \lambda (E_{11} + E_{22} + E_{33})^2 \right. \]

\[ + \mu (E_{11}^2 + E_{22}^2 + E_{33}^2) + 2\mu (E_{12}^2 + E_{13}^2 + E_{23}^2) \] \[ dV. \] (18)

Note that

\[ \frac{dV}{dt} = \int \int_V \int \left[ \rho_0 \frac{\partial^2 u_i}{\partial t^2} - (\lambda + \mu) \frac{\partial}{\partial x_i} + \mu \nabla^2 u_i \right] \frac{\partial u_i}{\partial t} dV. \] (19)

By using (15) we conclude that

\[ \frac{dV}{dt} = 0. \] (20)

We show next that \( V \) is positive definite with respect to \( \bar{\rho} \). Thus we have to show that there exists \( c > 0 \) such that \( V \geq c\bar{\rho} \). This inequality follows from

\[ V \geq \frac{1}{2} \int \int_V \int \left[ \lambda (E_{11} + E_{22} + E_{33})^2 \mu (E_{11}^2 + E_{22}^2 + E_{33}^2) \right. \]

\[ + 2\mu (E_{12}^2 + E_{13}^2 + E_{23}^2) \] \[ dV, \] (21)

and the estimate

\[ \frac{1}{2} \int \int_V \int \left[ \lambda (E_{11} + E_{22} + E_{33})^2 + \mu (E_{11}^2 + E_{22}^2 + E_{33}^2) \right. \]

\[ + 2\mu (E_{12}^2 + E_{13}^2 + E_{23}^2) \] \[ dV \geq \gamma^2 \int \int_V \int [u_1^2 + u_2^2 + u_3^2] dV. \] (22)

The value of \( \gamma^2 \) in (22) is estimated by the use of Friedrich's inequality (for more or less elementary means see Parton and Perlin (1984, vol II, p. 296)). By combining (21) and (22) we have

\[ V \geq \gamma^2 \bar{\rho}(u). \] (23)

It remains to show that the functional \( V(u, t) \) is continuous with respect to the metric \( \bar{\rho} \) at the time instant \( t = t_0 \in [0, T] \) and \( \bar{\rho}_0 = 0 \). To show this, note that

\[ V = \frac{1}{2} \int \int_V \int [\rho_0 \frac{\partial u_i}{\partial t} \frac{\partial u_i}{\partial t} + \lambda (E_{11} + E_{22} + E_{33})^2 \right. \]

\[ + \mu (E_{11}^2 + E_{22}^2 + E_{33}^2) + 2\mu (E_{12}^2 + E_{13}^2 + E_{23}^2) \] \[ dV < C\bar{\rho}_0^2, \] (24)
where $C$ is a constant depending on $\rho_0$, $\lambda$, $\mu$ and $V$ where $V$ is the volume of the body. Thus, all conditions stated in Theorem 1 of Section 10.3 are satisfied and we conclude that the equilibrium configuration (16) is Liapunov stable with respect to the metrics (17). From (23),(24) it follows that equations (13),(14) have a unique solution.

3. Stability of an elastic plate in a resisting medium, simply supported along all sides and subjected to uniformly distributed tangential follower forces

Consider an elastic plate as shown in Fig. 2. We assume that the plate is simply supported along all sides. The plate is loaded by uniformly distributed tangential load of intensity $q$ and its motion is resisted by linear viscous force $\beta(\partial W/\partial t)$, where $\beta$ is a constant.

We examine the stability of a trivial configuration of the plate in which the plate is plane. From equations (8.2-14) through (8.2-16) we find that the trivial configuration is

$$W^0 = 0; \quad Q^0_1 = Q^0_2 = S^0_{12} = T^0_2 = 0; \quad T^0_1 = -q(l - x_1). \quad (25)$$

We want to examine the stability of the configuration in which (25) holds. Let the perturbations of the variables $W, \ldots, T_1$, denoted by $\Delta W, \ldots, \Delta T_1$, be defined as $W = W_0 + \Delta W, \ldots, T_1 = T^0_1 + \Delta T_1$. Then, by using (25) and by neglecting the higher-order terms in perturbations we obtain from (8.2-14) through (8.2-16)

$$D \nabla^4 W + q(l - x_1) \frac{\partial^2 W}{\partial x_1 \partial x_1} + \beta \frac{\partial W}{\partial t} + \rho_0 \frac{\partial^2 W}{\partial t^2} = 0, \quad (26)$$
for $0 \leq x_1 \leq a$, $0 \leq x_2 \leq b$. In writing (26) we used as intensity of distributed forces ($q_3$ in equation (8.2-15)) the inertial force $\rho_0 \left( \frac{\partial^2 W}{\partial t^2} \right)$ and viscous friction force $\beta \left( \frac{\partial W}{\partial t} \right)$. Also in (26) we used $W$ for $\Delta W$. The boundary conditions corresponding to (26) are

$$W = 0, \quad \frac{\partial^2 W}{\partial x_1^2} = 0, \quad \text{for } x_1 = 0 \text{ and } x_1 = a;$$

$$W = 0, \quad \frac{\partial^2 W}{\partial x_2^2} = 0, \quad \text{for } x_2 = 0 \text{ and } x_2 = b. \quad (27)$$

We use Liapunov's second method to estimate the value of $q$ for which the configuration (25) is stable. Thus consider the functional $V$ and the metrics $\rho_0$ and $\bar{\rho}$ (see Knops and Wilkes (1973) and Leipholz (1987))

$$V = \int \int_A \left\{ \rho_0 \left( \frac{\partial W}{\partial t} \right)^2 + \frac{D}{2} \left[ (\nabla^2 W)^2 - \frac{q}{D} (a - x_1) \left( \frac{\partial W}{\partial x_1} \right)^2 \right] \right\} dA$$

$$+ \int \int_A \left\{ \frac{\beta^2}{4\rho_0} W^2 + \frac{\beta}{2} W \frac{\partial W}{\partial t} \right\} dA;$$

$$\bar{\rho}_0 = \left[ \sup_{x \in A} |W(x, t = 0)|^2 + \sup_{x \in A} |\nabla^2 W(x, t = 0)|^2 + \sup_{x \in V} \left| \frac{\partial W(x, t = 0)}{\partial t} \right|^2 \right]^{1/2};$$

$$\bar{\rho} = \left[ \int \int_A W^2 dA \right]^{1/2}, \quad (28)$$

where $A$ is the area of the middle plane of the plate. Differentiating various terms in (28) we obtain (see Leipholz (1987))

$$\frac{d}{dt} \int \int_A \rho_0 \left( \frac{\partial W}{\partial t} \right)^2 dA = \int \int_A \rho_0 \left( \frac{\partial^2 W}{\partial t^2} \right) \left( \frac{\partial W}{\partial t} \right) dA$$

$$\frac{d}{dt} \frac{D}{2} \int \int_A (\nabla^2 W)^2 dA = D \int \int_A (\nabla^2 W) \left( \nabla^2 \left( \frac{\partial W}{\partial t} \right) \right) dA$$

$$= D \int \int_A (\nabla^4 W) \left( \frac{\partial W}{\partial t} \right) dA;$$

$$\frac{d}{dt} \int \int_A \frac{q}{D} (a - x_1) \left( \frac{\partial W}{\partial x_1} \right)^2 dA = \int \int_A \frac{q}{D} (a - x_1) \left( \frac{\partial W}{\partial x_1} \right) \frac{\partial}{\partial t} \left( \frac{\partial W}{\partial x_1} \right) dA$$

$$= \frac{q}{D} \int \int_A \left[ (a - x_1) \left( \frac{\partial W}{\partial x_1} \right) \left( \frac{\partial W}{\partial t} \right) \right] dA.$$
where \( n \) is the unit normal vector to the boundary \( C \) and where we used boundary conditions (27). From (27) and (29) we obtain

\[
-a_1 \frac{d}{dt} \int \int \left\{ \frac{\beta^2}{4\rho_0} W^2 + \frac{\beta}{2} \frac{\partial W}{\partial t} \right\} dA
= \int \int \left\{ \frac{\beta^2}{2\rho_0} \frac{\partial W}{\partial t} + \frac{\beta}{2} \left( \frac{\partial W}{\partial t} \right)^2 + \frac{\beta}{2} W \frac{\partial^2 W}{\partial t^2} \right\} dA,
\]

where \( n \) is the unit normal vector to the boundary \( C \) and where we used boundary conditions (27). From (27) and (29) we obtain

\[
\frac{dV}{dt} = -\int \int \left\{ \frac{\beta}{2} \left( \frac{\partial W}{\partial t} \right)^2 + \frac{\beta D}{2\rho_0} (\nabla^2 W)^2 \right\} dA
- \frac{q\beta}{2\rho_0} (a - x_1) \left( \frac{\partial W}{\partial x_1} \right)^2 + q \left( \frac{\partial W}{\partial x_1} \right) \left( \frac{\partial W}{\partial t} \right) \right\} dA.
\]

The functional (28) is positive definite with respect to \( \bar{\rho} \) if the following two inequalities are satisfied

\[
\int \int \left\{ \frac{\rho_0}{2} \left( \frac{\partial W}{\partial t} \right)^2 + \frac{\beta^2}{4\rho_0} W^2 + \frac{\beta}{2} W \frac{\partial W}{\partial t} \right\} dA > 0;
\]

\[
\int \int \left\{ (\nabla^2 W)^2 - \frac{q}{D} (a - x_1) \left( \frac{\partial W}{\partial x_1} \right)^2 \right\} dA \geq c\bar{\rho}(W),
\]

with \( c > 0 \). The inequality (31)_1 is always satisfied since the integrand is the positive definite form of \( W \) and \((\partial W/\partial t)\) with the discriminant \( \beta/16 \).

We need now several inequalities to estimate \( c \) in (31)_2. First from the Wirtinger (or eigenvalue) inequality and boundary conditions \( W = 0 \) on the boundary of the plate \( C \) we have

\[
\int \int W^2 dA \leq c_1 \int \int \left[ \left( \frac{\partial W}{\partial x_1} \right)^2 + \left( \frac{\partial W}{\partial x_2} \right)^2 \right] dA,
\]

where

\[
\int \int \left( \frac{\partial W}{\partial x_1} \right)^2 + \left( \frac{\partial W}{\partial x_2} \right)^2 dA
= \int \int \left( \frac{\partial^2 W}{\partial x_1 \partial x_1} + \frac{\partial^2 W}{\partial x_2 \partial x_2} \right) dA
= \int \int \left( \frac{\partial^2 W}{\partial x_1 \partial x_1} \cos \theta(n, \bar{x}_1) dS \right) + \left( \frac{\partial^2 W}{\partial x_2 \partial x_2} \right) dA.
\]

The functional (28) is positive definite with respect to \( \bar{\rho} \) if the following two inequalities are satisfied

\[
\int \int \left\{ \frac{\rho_0}{2} \left( \frac{\partial W}{\partial t} \right)^2 + \frac{\beta^2}{4\rho_0} W^2 + \frac{\beta}{2} W \frac{\partial W}{\partial t} \right\} dA > 0;
\]

\[
\int \int \left\{ (\nabla^2 W)^2 - \frac{q}{D} (a - x_1) \left( \frac{\partial W}{\partial x_1} \right)^2 \right\} dA \geq c\bar{\rho}(W),
\]

with \( c > 0 \). The inequality (31)_1 is always satisfied since the integrand is the positive definite form of \( W \) and \((\partial W/\partial t)\) with the discriminant \( \beta/16 \).

We need now several inequalities to estimate \( c \) in (31)_2. First from the Wirtinger (or eigenvalue) inequality and boundary conditions \( W = 0 \) on the boundary of the plate \( C \) we have

\[
\int \int W^2 dA \leq c_1 \int \int \left[ \left( \frac{\partial W}{\partial x_1} \right)^2 + \left( \frac{\partial W}{\partial x_2} \right)^2 \right] dA,
\]

where

\[
\int \int \left( \frac{\partial W}{\partial x_1} \right)^2 + \left( \frac{\partial W}{\partial x_2} \right)^2 dA
= \int \int \left( \frac{\partial^2 W}{\partial x_1 \partial x_1} + \frac{\partial^2 W}{\partial x_2 \partial x_2} \right) dA
= \int \int \left( \frac{\partial^2 W}{\partial x_1 \partial x_1} \cos \theta(n, \bar{x}_1) dS \right) + \left( \frac{\partial^2 W}{\partial x_2 \partial x_2} \right) dA.
\]
where \( c_1 > 0 \). Next we use Poincaré's inequality (see Sobolev (1963))

\[
\int_A \int u^2 dA \leq c_3 \left\{ \int_A \int u dA \right\}^2 + \int_A \int \left[ \left( \frac{\partial u}{\partial x_1} \right)^2 + \left( \frac{\partial u}{\partial x_2} \right)^2 \right] dA
\]

applied to \( \partial W/\partial x_1 \) and \( \partial W/\partial x_2 \). From (33) we obtain

\[
\int_A \int \left\{ \left( \frac{\partial W}{\partial x_1} \right)^2 + \left( \frac{\partial W}{\partial x_2} \right)^2 \right\} dA
\]

\[
\leq c_3 \int_A \int \left\{ \left( \frac{\partial^2 W}{\partial x_1 \partial x_1} \right)^2 + 2 \left( \frac{\partial^2 W}{\partial x_1 \partial x_2} \right)^2 + \left( \frac{\partial^2 W}{\partial x_2 \partial x_2} \right)^2 \right\} dA,
\]

where we used the fact that

\[
\int_A \int \left( \frac{\partial W}{\partial x_1} \right) dA = \int_C W \cos \angle(n, \bar{x}_1) dS = 0;
\]

\[
\int_A \int \left( \frac{\partial W}{\partial x_2} \right) dA = \int_C W \cos \angle(n, \bar{x}_2) dS = 0.
\]

In (35) \( n \) is the unit normal vector to the boundary \( C \). We need now the following identity, that is, the consequence of the boundary conditions

\[
\int_A \int \left[ \left( \frac{\partial^2 W}{\partial x_1 \partial x_1} \right)^2 - \left( \frac{\partial^2 W}{\partial x_1 \partial x_2} \right) \left( \frac{\partial^2 W}{\partial x_2 \partial x_2} \right) \right] dA
\]

\[
= \int_A \int \left\{ \frac{\partial}{\partial x_1} \left[ \left( \frac{\partial^2 W}{\partial x_1 \partial x_2} \right) \left( \frac{\partial W}{\partial x_2} \right) \right] - \frac{\partial}{\partial x_2} \left[ \left( \frac{\partial^2 W}{\partial x_1 \partial x_1} \right) \left( \frac{\partial W}{\partial x_2} \right) \right] \right\} dA
\]

\[
= \int_C \left\{ \left( \frac{\partial^2 W}{\partial x_1 \partial x_2} \right) \left( \frac{\partial W}{\partial x_2} \right) \right\} \cos \angle(n, \bar{x}_1) - \left[ \left( \frac{\partial^2 W}{\partial x_1 \partial x_1} \right) \left( \frac{\partial W}{\partial x_2} \right) \right] \cos \angle(n, \bar{x}_2) dS = 0.
\]

Thus

\[
\int_A \int \left[ \left( \frac{\partial^2 W}{\partial x_1 \partial x_2} \right)^2 \right] dA = \int_A \int \left[ \left( \frac{\partial^2 W}{\partial x_1 \partial x_1} \right) \left( \frac{\partial^2 W}{\partial x_2 \partial x_2} \right) \right] dA.
\]

\(^3\)There are estimates for \((1/c_1)\). For example, \((\pi L/4A) < (1/c_1) < (\pi L/2A)\) where \( L \) is the length of the boundary curve \( C \) and \( A \) is its area (see Mitrinovic (1970)).
By using (37) we have

\[
\int \int_A (\nabla^2 W)^2 dA = \int \int_A \left\{ \left( \frac{\partial^2 W}{\partial x_1 \partial x_1} \right)^2 + \left( \frac{\partial^2 W}{\partial x_2 \partial x_2} \right)^2 + 2 \left( \frac{\partial^2 W}{\partial x_1 \partial x_2} \right) \left( \frac{\partial^2 W}{\partial x_2 \partial x_2} \right) \right\} dA
\]

\[
= \int \int_A \left\{ \left( \frac{\partial^2 W}{\partial x_1 \partial x_1} \right)^2 + \left( \frac{\partial^2 W}{\partial x_2 \partial x_2} \right)^2 + 2 \left( \frac{\partial^2 W}{\partial x_1 \partial x_2} \right)^2 \right\} dA.
\]  

(38)

The functional \( V \) given by (28) can be estimated as

\[
V = \int \int_A \left\{ \frac{\rho_0}{2} \left( \frac{\partial W}{\partial t} \right)^2 + \frac{D}{2} \left[ (\nabla^2 W)^2 - \frac{q}{D} (a - x_1) \left( \frac{\partial W}{\partial x_1} \right)^2 \right] \right\} dA
\]

\[
\geq \int \int_A \left\{ \frac{D}{2} \left[ (\nabla^2 W)^2 - \frac{q}{D} (a - x_1) \left( \frac{\partial W}{\partial x_1} \right)^2 \right] \right\} dA
\]

\[
\geq \frac{D}{2} \left( \frac{1}{c_3} - \frac{qa}{D} \right) \int \int_A \left[ \left( \frac{\partial W}{\partial x_1} \right)^2 + \left( \frac{\partial W}{\partial x_2} \right)^2 \right] dA
\]

\[
\geq \frac{D}{2} \left( \frac{1}{c_3} - \frac{qa}{D} \right) \frac{1}{c_1} \int \int_A W^2 dA = \frac{D}{2} \left( \frac{1}{c_3} - \frac{qa}{D} \right) \frac{1}{c_1} \bar{p}(W).
\]  

(39)

Therefore, if

\[
q < \frac{D}{c_3 a},
\]

(40)

\( V \) is positive definite with respect to \( \bar{p} \). We show next that \( V \) is continuous with respect to \( \bar{\rho}_0 \) at \( t_0 = 0 \) and \( \bar{\rho}_0 = 0 \). Thus by using (38) in (28) we obtain

\[
V = \int \int_A \left\{ \frac{\rho_0}{2} \left( \frac{\partial W}{\partial t} \right)^2 + \frac{D}{2} \left[ (\nabla^2 W)^2 - \frac{q}{D} (a - x_1) \left( \frac{\partial W}{\partial x_1} \right)^2 \right] \right\} dA
\]

\[
+ \int \int_A \left\{ \frac{\beta^2}{4\rho_0} W^2 + \frac{\beta}{2} W \frac{\partial W}{\partial t} \right\} dA
\]

\[
\leq \int \int_A \left\{ \frac{\rho_0}{2} \left( \frac{\partial W}{\partial t} \right)^2 + \frac{D}{2} \left[ (\nabla^2 W)^2 \right] \right\} dA
\]

\[
+ \int \int_A \left\{ \frac{\beta^2}{4\rho_0} W^2 + \frac{\beta}{2} W \frac{\partial W}{\partial t} \right\} dA
\]
However, since
\[
\frac{\rho_0}{2} \left( \frac{\partial W}{\partial t} \right)^2 + \frac{\beta}{2} W \left( \frac{\partial W}{\partial t} \right) + \frac{\beta^2}{4 \rho_0} W^2 < \rho_0 \left( \frac{\partial W}{\partial t} \right)^2 + \frac{\beta^2}{2 \rho_0} W^2, \tag{42}
\]
we have
\[
V \leq \int \int_A \left\{ \rho_0 \left( \frac{\partial W}{\partial t} \right)^2 + \frac{\beta^2}{2 \rho_0} W^2 + \frac{D}{2} \left[ \left( \frac{\partial^2 W}{\partial x_1 x_1} \right)^2 + \left( \frac{\partial^2 W}{\partial x_2 x_2} \right)^2 + 2 \left( \frac{\partial^2 W}{\partial x_1 \partial x_2} \right)^2 \right] \right\} dA
\leq (\tilde{\rho}_0)^2 A c^*, \tag{43}
\]
where $A = ab$ is the area of the plate and $c^*$ is a constant depending on $\rho_0$, $\beta^2$ and $D$. It remains now to show that $V$ is nonincreasing. From (28) we have
\[
-\int \int_A \left\{ W \frac{\partial}{\partial x_1} \left[ (a - x_1) \frac{\partial W}{\partial x_1} \right] \right\} dA = \int \int_A \left\{ (a - x_1) \left( \frac{\partial W}{\partial x_1} \right)^2 \right\} dA
+ \int_C W(a - x_1) \cos \angle(n, \bar{x}) dA = \int \int_A \left\{ (a - x_1) \left( \frac{\partial W}{\partial x_1} \right)^2 \right\} dA, \tag{44}
\]
where we used the boundary conditions. Let $q_{cr}$ be the value of Rayleigh's quotient for the case when the distributed load is conservative; that is,
\[
q_{cr} = \min \frac{\int \int_A u(\nabla^2 u)^2 dA}{-\int \int_A \left\{ u \frac{\partial}{\partial x_1} \left[ (a - x_1) \left( \frac{\partial u}{\partial x_1} \right) \right] \right\} dA}, \tag{45}
\]
where \( u \) is an admissible function, that is, a function satisfying boundary conditions (27) and certain regularity conditions. From (44) we obtain

\[
q_{cr} \leq \frac{\int \int_A W(\nabla^2 W)^2 \, dA}{\int \int_A \left\{ W(a - x_1) \left( \frac{\partial W}{\partial x_1} \right)^2 \right\} \, dA}.
\]  

(46)

Finally we use the mean value theorem for integrals that guarantees the existence of a positive constant \( m \) such that

\[
\int \int_A \left\{ (a - x_1) \left( \frac{\partial W}{\partial x_1} \right)^2 \right\} \, dA = m \int \int_A \left( \frac{\partial W}{\partial x_1} \right)^2 \, dA, \quad 0 \leq m \leq a. \]  

(47)

By using (46),(47) in (30) we have

\[
\frac{dV}{dt} \leq -\int \int_A \left\{ \beta \left( \frac{\partial W}{\partial t} \right)^2 + \frac{\beta m}{2 \rho_0} (q - q_{cr}) \left( \frac{\partial W}{\partial x_1} \right)^2 + q \left( \frac{\partial W}{\partial x_1} \right) \left( \frac{\partial W}{\partial t} \right) \right\} \, dA.
\]  

(48)

The integrand in (48) is the positive definite form of \( (\partial W/\partial t) \) and \( (\partial W/\partial x_1) \) if its determinant is positive

\[
\frac{\beta}{2} \left[ \frac{\beta m}{2 \rho_0} (q_{cr} - q) \right] - \frac{q^2}{4} > 0.
\]  

(49)

From (49) we conclude that

\[
\beta > q \left[ \frac{\rho_0}{m(q_{cr} - q)} \right]^{1/2},
\]  

(50)

implies \((dV/dt) < 0\). Therefore according to Theorem 1 of Section 10.3 the trivial configuration of the plate described by (25) is Liapunov stable with respect to metrics given by (31) if the inequalities (41) and (50) are satisfied.

There are various methods to obtain constants in the inequalities used in estimating the stability boundary (40). For example, for inequalities of type (32) consult Weinstein (1937) and Pólya and Szegö (1951).

4. Stability of a rotating elastic plate positioned in a rigid cylinder

Consider an elastic plate, shown in Fig. 3, fixed to a rigid tube that is rotating about its geometrical axis with constant angular velocity \( \omega \).
From the results of Problem 7 in Chapter 6 we have that the relevant equations for the stress field in the trivial configuration in which plate is plane are

\[
\begin{align*}
\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{1}{r} (\sigma_{rr} - \sigma_{\theta \theta}) + f_r &= 0; \\
\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta \theta}}{\partial \theta} + \frac{2}{r} \sigma_{r\theta} + f_\theta &= 0; \\
E_{rr} &= \frac{1}{E} (\sigma_{rr} - \nu \sigma_{\theta \theta}); \\
E_{\theta \theta} &= \frac{1}{E} (\sigma_{\theta \theta} - \nu \sigma_{rr}); \\
E_{r\theta} &= \frac{1 + \nu}{E} \sigma_{r\theta}. 
\end{align*}
\] (51)

The boundary conditions corresponding to the plate shown in Fig. 3 are

\[
\begin{align*}
u_r(r = 0) &= 0; \\
u_r(r = R) &= 0. 
\end{align*}
\] (52)

By using the same procedure as in the problem in Section 5.3 we obtain the solution of the form (5.3-6) which has to satisfy the boundary conditions (52). Omitting the details, we write the solution for the stress component \( \sigma_{rr} \) in the trivial state as

\[
\sigma_{rr}^0 = \rho_0 \omega^2 R^2 \left[ \frac{1 + \nu}{8} - \frac{3 + \nu}{8} \left( \frac{r}{R} \right)^2 \right]. 
\] (53)

The differential equation of superimposed motion is obtained if \( W \) is assumed to be a function of \( r \) and the time \( t \). Then, by using equation (c) of Problem 2 in Chapter 8, we obtain

\[
D \frac{1}{r} \frac{\partial}{\partial r} \left\{ r \frac{\partial}{\partial r} \left( r \frac{\partial \rho_0 \omega^2 \rho^2 W}{\partial \rho} \right) \right\} = -\rho_0 h \frac{\partial^2 W}{\partial t^2} + \frac{1}{r} \frac{\partial}{\partial r} \left( r h \sigma_{rr}^0 \frac{\partial \rho_0 \omega^2 \rho^2 W}{\partial \rho} \right), 
\] (54)

subject to

\[
W(r = R, t) = 0; \quad \frac{\partial W(r = R, t)}{\partial r} = 0. 
\] (55)

In writing (54) we used the fact that \( T_r = h \sigma_{rr}^0 \), and that the only distributed force is the inertial force; that is, \( q_3 = -\rho_0 h (\partial^2 W / \partial t^2) \). Next we introduce dimensionless quantities

\[
\lambda = \frac{\rho_0 \omega^2 R^4}{E h^3}; \quad y = \frac{W}{R}; \quad x = \frac{r}{R}; \quad \tau = t \sqrt{\frac{D}{h \rho_0 R^4}}. 
\] (56)
By using (53) and (56) in (54), (55) we obtain
\[
\frac{1}{x} \frac{\partial}{\partial x} \left\{ \frac{1}{x} \frac{\partial}{\partial x} \left[ \frac{1}{x} \frac{\partial}{\partial x} \left( x \frac{\partial y}{\partial x} \right) \right] \right\} + \frac{\partial^2 y}{\partial x^2} = 0,
\]
\[-12(1 - \nu^2) \frac{1}{x} \frac{\partial}{\partial x} \left[ x \left( \frac{1 + \nu}{8} - \frac{3 + \nu}{8} x^2 \right) \frac{\partial y}{\partial x} \right] = 0, \tag{57}\]
\[y(x = 1, \tau) = 0; \quad \frac{\partial y(x = 1, \tau)}{\partial x} = 0. \tag{58}\]

To solve (57), (58) we use the Galerkin method (see Vujanovic and Jones (1989)). Thus, we assume
\[y(x, \tau) = \sum_{i=0}^{N} C_i R_i(x) \exp(i\Omega \tau), \tag{59}\]
where \(C_i\) are constants, \(R_i\) are specified functions, and the frequency \(\Omega\) is to be determined. It is the clear form (59) that real \(\Omega\) guarantees stability. We take \(R_i\) as
\[R_i = [1 - 2x^2 + x^4] x^{2i}, \tag{60}\]
and with \(N = 5\) in (59). By substituting (59) into (57) and by multiplying the result by \(x R_i(x)\) we obtain, after integration,
\[\sum_{i=0}^{N} C_i (A_{ij} - \Omega^2 B_{ij}) = 0, \tag{61}\]
where
\[A_{ij} = \int_0^1 \left\{ \frac{1}{x} \frac{d}{dx} \left\{ \frac{1}{x} \frac{d}{dx} \left( \frac{dR_i}{dx} \right) \right\} \right\} \frac{dR_j}{dx}; \tag{62}\]
\[B_{ij} = \int_0^1 R_i R_j x dx. \tag{62}\]

The nontrivial constants \(C_i\) exist if the following frequency equation is satisfied
\[\det |A_{ij} - \Omega^2 B_{ij}| = 0. \tag{63}\]

From the condition (63) the numerical values of \(\Omega^2\) are determined in terms of \(\lambda\) (see Maretic (1998) where the different boundary conditions are treated too). For the case of boundary conditions (53) the critical value of \(\lambda\) (the value for which \(\Omega = 0\)) for \(\nu = 0.3\) is found to be
\[\lambda_{cr} = 38.8533. \tag{64}\]
We make a remark concerning the eigenmodal analysis used in this example. The condition $\lambda < \lambda_{cr}$ guarantees that in the first vibration mode the plate will not oscillate with increasing amplitude and that for $\lambda > \lambda_{cr}$ at least for one initial condition (corresponding to the first mode) the plate will oscillate with increasing amplitude and therefore it is unstable. It remains, however, to show that all solutions of (57),(58) can be expressed in the form (59) and that the series (59) is convergent. In the engineering applications this is usually omitted and (64) is taken as the critical dimensionless rotation speed.

5. **Stability of a plate with shear stresses on its sides**

Consider a plate as shown in Fig. 4. The plate is loaded by shearing forces of intensity $\tau$ (per unit length of the plate side) along all its sides.

Assume further that the plate is simply (freely) supported along its sides so that the boundary conditions read (see (8.3-3))

$$W(x_1 = 0, x_2) = \frac{\partial^2 W(x_1 = 0, x_2)}{\partial x_1^2} = 0;$$

$$W(x_1 = a, x_2) = \frac{\partial^2 W(x_1 = a, x_2)}{\partial x_1^2} = 0;$$

$$W(x_1, x_2 = 0) = \frac{\partial^2 W(x_1, x_2 = 0)}{\partial x_2^2} = 0;$$

$$W(x_1, x_2 = b) = \frac{\partial^2 W(x_1, x_2 = b)}{\partial x_2^2} = 0. \quad (65)$$

We use the energy method for stability analysis. The potential energy of inner forces is (see Volmir (1967, p.321))

$$\Pi^i = \frac{1}{2}D \int_A \int \left\{ (\nabla^2 W)^2 - (1 - \nu) L(W,W) \right\} dA, \quad (66)$$
where $A$ is the surface of the plate and

$$\nabla^2 W = \frac{\partial^2 W}{\partial x_1^2} + \frac{\partial^2 W}{\partial x_2^2}, \quad L(W, W) = 2 \left[ \frac{\partial^2 W}{\partial x_1^2} \frac{\partial^2 W}{\partial x_2^2} - \left( \frac{\partial^2 W}{\partial x_1 \partial x_2} \right)^2 \right]. \quad (67)$$

The work of the outer forces consists of the work of shearing forces $\tau$ on the displacements of the boundaries of the plate. The potential energy is equal to this work with negative sign. It could be shown that (see Volmir (1967 p. 324)) the potential energy is

$$\Pi^o = -\tau \int \int_A \left\{ \frac{\partial W}{\partial x_1} \frac{\partial W}{\partial x_2} \right\} dA. \quad (68)$$

Therefore for the total potential energy of the system we have

$$\Pi = \frac{1}{2} D \int \int_A \left\{ (\nabla^2 W)^2 - (1 - \nu)L(W, W) - \tau \frac{\partial W}{\partial x_1} \frac{\partial W}{\partial x_2} \right\} dA. \quad (69)$$

We use the Ritz method (see Section 4.6) to determine the critical value of $\tau$. Thus, we assume $W$ in the form

$$W = \sum_{m=1}^{2} \sum_{n=1}^{2} A_{mn} \sin \frac{m \pi x_1}{a} \sin \frac{n \pi x_2}{b}, \quad (70)$$

where $A_{mn}$ are constants. The function (70) satisfies the boundary conditions (65). By substituting (70) into (69) and by differentiating the result with respect to $A_{mn}$ (the necessary condition for the minimum of $\Pi$) we obtain

$$\pi^4 D A_{11} \left( \frac{1}{a^2} + \frac{1}{b^2} \right)^2 - \frac{32}{9} \tau A_{22} = 0;$$

$$\frac{32}{9} \tau A_{11} - 4 \pi^4 D a b A_{22} \left( \frac{1}{a^2} + \frac{1}{b^2} \right)^2 = 0. \quad (71)$$

Let

$$\alpha = \frac{a}{b}; \quad \lambda = \frac{\pi^4 D}{32 \alpha b^4}. \quad (72)$$

With (72) the condition that (71) has a nontrivial solution becomes

$$\det \begin{vmatrix} \frac{4 \pi^4}{a^2} (1 + \alpha^2)^2 & 4 \frac{9}{\alpha^2} \left( \frac{16 \lambda}{a^2} (1 + \alpha^2)^2 \right) \end{vmatrix} = 0. \quad (73)$$

From (73) it follows that

$$\lambda = \pm \frac{\alpha^2}{9(1 + \alpha^2)^2}, \quad (74)$$
so that the critical shearing force (per unit length of the plate side) is

\[ \tau_{cr} = \frac{9\pi^4 D(1 + \alpha^2)^2}{32ba^3b^2}. \]  

(75)

We close this example with two comments. First, the value (75) could be improved by using more terms in the expansion (70). This leads to more complicated systems than (71). Second, we have to prove that \( \Pi(A_{mn}, \lambda) \) is in minimum for \( \lambda = \lambda_{cr} \). We omit this analysis.

**Problems**

1. For the plate shown in the figure, simply supported on all sides, use the von Kármán plate theory to determine the critical load \( q \) according to the Euler method.

   ![Diagram of a plate with loads](image)

   a) Show that equations (8.2-20),(8.2-22) for small deformations superimposed on the trivial configuration in which the plate is plane and loaded at its ends by \( q \) is

   \[ D \nabla^4 W + q \frac{\partial^2 W}{\partial x_1^2} = 0, \]  

   (a) and that the boundary conditions corresponding to (a) are

   \[ W = 0; \quad \frac{\partial^2 W}{\partial x_1^2} = 0; \quad \text{for } x_1 = 0 \quad \text{and } x_1 = a, \]  

   \[ W = 0; \quad \frac{\partial^2 W}{\partial x_2^2} = 0; \quad \text{for } x_2 = 0 \quad \text{and } x_2 = b. \]  

   (b)

   b) Assume solution of (a),(b) in the from

   \[ W = C \sin \frac{m\pi x_1}{a} \sin \frac{n\pi x_2}{b}, \]  

   (c)
where \( C \) is constant and \( m \) and \( n \) are integers. By using (a) show that
\[
q = \frac{D\pi^2}{b^2} \left( \frac{mb}{a} + \frac{n^2a}{mb} \right)^2. \tag{d}
\]
The critical load is then determined as
\[
q_{cr} = \min_{m,n} \left[ \frac{D\pi^2}{b^2} \left( \frac{mb}{a} + \frac{n^2a}{mb} \right)^2 \right]. \tag{e}
\]
The complete bifurcation analysis showing that von Kármán plate equations have a bifurcation point for the compressive load (e) is presented in Chow and Hale (1982).

2. Consider an equilibrium configuration
\[
u = 0 \tag{a}
\]
of an elastic body. Consider its stability for irrotational wave motion propagating in the \( \bar{x}_1 \) direction (see (5.14-5)) so that \((u_1 = u, u_2 = u_3 = 0)\),
\[
(\lambda + \mu) \frac{\partial^2 u}{\partial x^2} = \rho_0 \frac{\partial^2 u}{\partial t^2}, \tag{b}
\]
subject to
\[
u = 0 \quad \text{on} \quad S. \tag{c}
\]
By using the total energy as the Liapunov functional
\[
V = \frac{1}{2} \int_V \int \left[ \rho_0 \left( \frac{\partial u}{\partial t} \right)^2 + (\lambda + 2\mu) \left( \frac{\partial u}{\partial x} \right)^2 \right] \, dV, \tag{d}
\]
and the inequality (see Knops and Wilkes (1966))
\[
\int \int_V \int u^2 \, dV \leq c \int \int_V \int \left( \frac{\partial u}{\partial x} \right)^2 \, dV, \tag{e}
\]
where \( c > 0 \), show that (a) is stable with respect to the metrics
\[
\tilde{\rho}_0 = \sup_x \left| \frac{\partial u(0, x)}{\partial x} \right| + \sup_x \left| \frac{\partial u(0, x)}{\partial t} \right|;
\]
\[
\tilde{\rho} = \int \int_V \int \left[ u^2 + \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial t} \right)^2 \right] \, dV. \tag{f}
\]
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